Journal of Number Theory 133(2013), 1572-1595.

CONGRUENCES INVOLVING $\binom{2k}{k}^2\binom{3k}{k}$

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Abstract. Let $p>3$ be a prime, and let m be an integer with $p\nmid m$. In this paper, based on the work of Brillhart and Morton, by using the work of Ishii and Deuring's theorem for elliptic curves with complex multiplication we solve some conjectures of Zhi-Wei Sun concerning $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/m^k \pmod{p^2}$.

MSC: Primary 11A07, Secondary 33C45, 11E25, 11G07, 11L10, 05A10, 05A19
Keywords: Congruence; Legendre polynomial; character sum; elliptic curve; binary quadratic form

1. Introduction.

For positive integers a,b and n, if $n=ax^2+by^2$ for some integers x and y, we briefly say that $n=ax^2+by^2$. Let $p>3$ be a prime. In 2003, Rodriguez-Villegas[RV] posed some conjectures on supercongruences modulo p^2. One of his conjectures is equivalent to

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

This conjecture has been solved by Mortenson[Mo] and Zhi-Wei Sun[Su4].

Let $\mathbb{Z}$ be the set of integers, and for a prime p let R_p be the set of rational numbers whose denominator is coprime to p. Recently the author's brother Zhi-Wei Sun[Su1] posed many conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}m^{-k} \pmod{p^2}$, where $p>3$ is a prime and $m\in\mathbb{Z}$ with $p\nmid m$. For example, he conjectured that (see [Su1, Conjecture A13])

$$(1.1)\qquad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 7,11,13,14 \pmod{15},\\ 4x^2-2p \pmod{p^2} & \text{if } p=x^2+15y^2\equiv 1,4 \pmod{15},\\ 20x^2-2p \pmod{p^2} & \text{if } p=5x^2+3y^2\equiv 2,8 \pmod{15}.\end{cases}$$

Let $\{P_n(x)\}$ be the Legendre polynomials given by (see [MOS, pp. 228-232], [G, (3.132)-(3.133)])

$$(1.2)\qquad P_n(x)=\frac{1}{2^n}\sum_{k=0}^{[n/2]}\binom{n}{k}(-1)^k\binom{2n-2k}{n}x^{n-2k}=\frac{1}{2^n\cdot n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n,$$

where $[a]$ is the greatest integer not exceeding a. From (1.2) we see that

$$P_n(-x) = (-1)^n P_n(x). \tag{1.3}$$

Let $p > 3$ be a prime. Then $\binom{2k}{k}\binom{3k}{k} = \frac{(3k)!}{k!^3} \equiv 0 \pmod p$ for $\frac p3 < k < p$. In [S2] the author showed that

$$P_{[\frac p3]}(t) \equiv \sum_{k=0}^{[p/3]} \binom{2k}{k}\binom{3k}{k}\Big(\frac{1-t}{54}\Big)^k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}\binom{3k}{k}\Big(\frac{1-t}{54}\Big)^k \pmod p. \tag{1.4}$$

In the paper, using the work of Brillhart and Morton[BM] we prove that

$$P_{[\frac p3]}(t) \equiv -\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big) \pmod p, \tag{1.5}$$

where $(\frac ap)$ is the Legendre symbol. Based on (1.5) and the work of Ishii[I], we determine

$$P_{[\frac p3]}(t) \pmod p \quad \text{for} \quad t = \frac54, \frac{5}{\sqrt{-2}}, \frac{\sqrt{-11}}{4}, \frac{1}{\sqrt2}, \sqrt5, \frac{9}{20}\sqrt5, \frac{\sqrt{17}}{4}, \frac{5}{32}\sqrt{41}, \frac{53}{500}\sqrt{89}.$$

For instance, if $p \equiv 1, 4 \pmod 5$ is a prime, we prove that

$$P_{[\frac p3]}(\sqrt5) \equiv \begin{cases} 2x(\frac x3) \pmod p & \text{if } p = x^2+15y^2 \equiv 1, 4 \pmod{15}, \\ 0 \pmod p & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases}$$

Let $p > 3$ be a prime, $m \in R_p$, $m \not\equiv 0 \pmod p$ and $t = \sqrt{1-108/m}$. In the paper we show that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k} \equiv P_{[\frac p3]}(t)^2 \pmod p \tag{1.6}$$

and that

$$P_{[\frac p3]}(t) \equiv 0 \pmod p \quad \text{implies} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k} \equiv 0 \pmod{p^2}. \tag{1.7}$$

On the basis of (1.6) and (1.7), we prove some congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}m^{-k}$ in the cases $m = 8, 64, 216, -27, -192, -8640, -12^3, -48^3, -300^3$. Thus we partially solve some conjectures posed by Zhi-Wei Sun in [Su1,Su2,Su3]. As a typical example, for odd primes $p \neq 11$ we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k} \equiv \begin{cases} x^2 \pmod p & \text{if } (\frac{p}{11}) = 1 \text{ and so } 4p = x^2+11y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1. \end{cases}$$

2. A general congruence modulo p^2.

We begin with a useful combinatorial identity.

Lemma 2.1. *For any nonnegative integer n we have*

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{3k}{k}\binom{k}{n-k}(-27)^{n-k}=\sum_{k=0}^{n}\binom{2k}{k}\binom{3k}{k}\binom{2(n-k)}{n-k}\binom{3(n-k)}{n-k}.$$

Proof. Let m be a nonnegative integer. For $k\in\{0,1,\ldots,m\}$ set

$$\begin{aligned}F_1(m,k)&=\binom{2k}{k}^2\binom{3k}{k}\binom{k}{m-k}(-27)^{m-k},\\ F_2(m,k)&=\binom{2k}{k}\binom{3k}{k}\binom{2(m-k)}{m-k}\binom{3(m-k)}{m-k}.\end{aligned}$$

For $k\in\{0,1,\ldots,m+1\}$ set

$$\begin{aligned}G_1(m,k)&=-\frac{(27k)^2(m+2)(m-2k)(m-2k+1)}{(k+1)(k+2)}\binom{k+2}{m+2-k}\binom{2k}{k}^2\binom{3k}{k}(-27)^{m-k},\\ G_2(m,k)&=\frac{3k^2(9m^2-9mk+30m-14k+24)}{(m+2-k)^2}\\ &\quad\times\binom{2k}{k}\binom{3k}{k}\binom{2(m+1-k)}{m+1-k}\binom{3(m+1-k)}{m+1-k}.\end{aligned}$$

For $i=1,2$ and $k\in\{0,1,\ldots,m\}$, it is easy to check that

$$\begin{aligned}&(m+2)^3F_i(m+2,k)-3(2m+3)(9m^2+27m+22)F_i(m+1,k)\\ &\quad+81(m+1)(3m+2)(3m+4)F_i(m,k)=G_i(m,k+1)-G_i(m,k).\end{aligned}\tag{2.1}$$

Set $S_i(n)=\sum_{k=0}^{n}F_i(n,k)$ for $n=0,1,2,\ldots$. Then

$$\begin{aligned}&(m+2)^3(S_i(m+2)-F_i(m+2,m+2)-F_i(m+2,m+1))\\ &\quad-3(2m+3)(9m^2+27m+22)(S_i(m+1)-F_i(m+1,m+1))\\ &\quad+81(m+1)(3m+2)(3m+4)S_i(m)\\ &=(m+2)^3\sum_{k=0}^{m}F_i(m+2,k)-3(2m+3)(9m^2+27m+22)\sum_{k=0}^{m}F_i(m+1,k)\\ &\quad+81(m+1)(3m+2)(3m+4)\sum_{k=0}^{m}F_i(m,k)\\ &=\sum_{k=0}^{m}(G_i(m,k+1)-G_i(m,k))=G_i(m,m+1)-G_i(m,0)=G_i(m,m+1).\end{aligned}$$

Observe that

$$\begin{aligned}
F_1(m+1,m+1) &= \binom{2m+2}{m+1}^2\binom{3m+3}{m+1},\\
F_1(m+2,m+1) &= -27(m+1)\binom{2m+2}{m+1}^2\binom{3m+3}{m+1},\\
F_1(m+2,m+2) &= \frac{6(2m+3)(3m+4)(3m+5)}{(m+2)^3}\binom{2m+2}{m+1}^2\binom{3m+3}{m+1},\\
F_2(m+1,m+1) &= \binom{2m+2}{m+1}\binom{3m+3}{m+1},\\
F_2(m+2,m+1) &= 6\binom{2m+2}{m+1}\binom{3m+3}{m+1},\\
F_2(m+2,m+2) &= \frac{3(3m+4)(3m+5)}{(m+2)^2}\binom{2m+2}{m+1}\binom{3m+3}{m+1}
\end{aligned}$$

and

$$\begin{aligned}
G_1(m,m+1) &= 27(m+1)^3(m+2)\binom{2m+2}{m+1}^2\binom{3m+3}{m+1},\\
G_2(m,m+1) &= 3(m+1)^2(7m+10)\binom{2m+2}{m+1}\binom{3m+3}{m+1}.
\end{aligned}$$

From the above we deduce that for $i=1,2$ and $m=0,1,2,\ldots,$

$$\begin{aligned}
&(m+2)^3S_i(m+2)-3(2m+3)(9m^2+27m+22)S_i(m+1)\\
&\quad+81(m+1)(3m+2)(3m+4)S_i(m)\\
&=G_i(m,m+1)+(m+2)^3(F_i(m+2,m+2)+F_i(m+2,m+1))\\
&\quad-3(2m+3)(9m^2+27m+22)F_i(m+1,m+1)=0.
\end{aligned}\tag{2.2}$$

Since $S_1(0)=1=S_2(0)$ and $S_1(1)=12=S_2(1)$, from (2.2) we deduce $S_1(n)=S_2(n)$ for all $n=0,1,2,\ldots$. This completes the proof.

Remark 2.1 We actually find (2.1) and prove Lemma 2.1 by using WZ method and Maple. The author thanks Professor Qing-Hu Hou for his help in finding (2.1). For the WZ method, see [PWZ].

Theorem 2.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}(x(1-27x))^k\equiv\Big(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}x^k\Big)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}(x(1-27x))^k\\
&=\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}x^k\sum_{r=0}^{k}\binom{k}{r}(-27x)^r\\
&=\sum_{m=0}^{2(p-1)}x^m\sum_{k=0}^{min\{m,p-1\}}\binom{2k}{k}^2\binom{3k}{k}\binom{k}{m-k}(-27)^{m-k}.
\end{aligned}$$

Suppose $p\le m\le 2p-2$ and $0\le k\le p-1$. If $k>\frac p2$, then $p\mid\binom{2k}{k}$ and so $p^2\mid\binom{2k}{k}^2$. If $k<\frac p2$, then $m-k\ge p-k>k$ and so $\binom{k}{m-k}=0$. Thus, from the above and Lemma 2.1 we deduce that

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}(x(1-27x))^k\\
&\equiv\sum_{m=0}^{p-1}x^m\sum_{k=0}^{m}\binom{2k}{k}^2\binom{3k}{k}\binom{k}{m-k}(-27)^{m-k}\\
&=\sum_{m=0}^{p-1}x^m\sum_{k=0}^{m}\binom{2k}{k}\binom{3k}{k}\binom{2(m-k)}{m-k}\binom{3(m-k)}{m-k}\\
&=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}x^k\sum_{m=k}^{p-1}\binom{2(m-k)}{m-k}\binom{3(m-k)}{m-k}x^{m-k}\\
&=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}x^k\sum_{r=0}^{p-1-k}\binom{2r}{r}\binom{3r}{r}x^r\\
&=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}x^k\Big(\sum_{r=0}^{p-1}\binom{2r}{r}\binom{3r}{r}x^r-\sum_{r=p-k}^{p-1}\binom{2r}{r}\binom{3r}{r}x^r\Big)\\
&=\Big(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}x^k\Big)^2-\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}x^k\sum_{r=p-k}^{p-1}\binom{2r}{r}\binom{3r}{r}x^r \pmod{p^2}.
\end{aligned}$$

If $\frac{2p}{3}\le k\le p-1$, then $\binom{2k}{k}\binom{3k}{k}=\frac{(3k)!}{k!^3}\equiv 0\pmod{p^2}$. If $0\le k\le\frac p3$ and $p-k\le r\le p-1$, then $\frac{2p}{3}\le r\le p-1$ and so $\binom{2r}{r}\binom{3r}{r}=\frac{(3r)!}{r!^3}\equiv 0\pmod{p^2}$. If $\frac p3<k<\frac{2p}{3}$ and $p-k\le r\le p-1$, then $r\ge p-k>\frac p3$, $\binom{2k}{k}\binom{3k}{k}=\frac{(3k)!}{k!^3}\equiv 0\pmod p$ and $\binom{2r}{r}\binom{3r}{r}=\frac{(3r)!}{r!^3}\equiv 0\pmod p$. Hence, for $0\le k\le p-1$ and $p-k\le r\le p-1$ we have $p^2\mid\binom{2k}{k}\binom{3k}{k}\binom{2r}{r}\binom{3r}{r}$ and so

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}x^k\sum_{r=p-k}^{p-1}\binom{2r}{r}\binom{3r}{r}x^r\equiv 0\pmod{p^2}.$$

Thus the result follows.

Corollary 2.1. *Let $p>3$ be a prime and $m\in R_p$ with $m\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}\equiv\Big(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\Big(\frac{1-\sqrt{1-108/m}}{54}\Big)^k\Big)^2 \pmod{p^2}.$$

Proof. Taking $x=\frac{1-\sqrt{1-108/m}}{54}$ in Theorem 2.1 we deduce the result.

3. A congruence for $P_{[p/3]}(t) \pmod p$.

Let $W_n(x)$ be the Deuring polynomial given by

$$W_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2x^k. \tag{3.1}$$

It is known that ([G,(3.134)],[BM])

$$W_n(x)=(1-x)^nP_n\Big(\frac{1+x}{1-x}\Big). \tag{3.2}$$

Let $p>3$ be a prime , $m,n\in R_p$ and $4m^3+27n^2\not\equiv 0 \pmod p$. From [Mor, Theorem 3.3] we have

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\ &\equiv -(-48m)^{\frac{1-(\frac p3)}{2}}(864n)^{\frac{1-(\frac{-1}{p})}{2}}(-16(4m^3+27n^2))^{[\frac{p}{12}]}J_p\Big(\frac{2^8\cdot 3^3m^3}{4m^3+27n^2}\Big) \pmod p,\end{aligned} \tag{3.3}$$

where $J_p(t)$ is a certain Jacobi polynomial given by

$$J_p(t)=1728^{[\frac{p}{12}]}P_{[\frac{p}{12}]}^{(-\frac13(\frac p3),-\frac12(\frac{-1}{p}))}\Big(1-\frac{t}{864}\Big) \tag{3.4}$$

and

$$P_k^{(\alpha,\beta)}(x)=\frac{1}{2^k}\sum_{r=0}^{k}\binom{k+\alpha}{r}\binom{k+\beta}{k-r}(x-1)^{k-r}(x+1)^r.$$

Theorem 3.1. *Let $p>3$ be a prime and $t\in R_p$. Then*

$$P_{[\frac p3]}(t)\equiv-\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big) \pmod p.$$

Proof. It is well known that $P_n(1)=1$. Since $P_{[\frac p3]}(1)=1$ and

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-3x-2}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{(x+1)^2(x-2)}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{x-2}{p}\Big)-\Big(\frac{-1-2}{p}\Big)=-\Big(\frac p3\Big),$$

the result is true for $t\equiv 1\pmod p$. As $P_{[\frac p3]}(-1)=(-1)^{[\frac p3]}P_{[\frac p3]}(1)=(\frac p3)$ and

$$\begin{aligned}\sum_{x=0}^{p-1}\Big(\frac{x^3-27x+54}{p}\Big)&=\sum_{x=0}^{p-1}\Big(\frac{(-3x)^3-27(-3x)+54}{p}\Big)\\&=\Big(\frac{-3}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3x-2}{p}\Big)=-1,\end{aligned}$$

the result is also true for $t\equiv -1\pmod p$.

Now we assume $t\not\equiv \pm1\pmod p$. Set $W_n(x)=\sum_{k=0}^n\binom nk^2x^k$. From [BM, Theorem 6] we know that

$$W_{[\frac p3]}\Big(1-\frac{x}{27}\Big)\equiv u_p(x)(x-27)^{[\frac p{12}]}J_p\Big(\frac{x(x-24)^3}{x-27}\Big)\pmod p, \tag{3.5}$$

where $J_p(x)$ is given by (3.4) and

$$u_p(x)=\begin{cases}1 & \text{if } p\equiv 1\pmod{12},\\ -3(x-24) & \text{if } p\equiv 5\pmod{12},\\ x^2-36x+216 & \text{if } p\equiv 7\pmod{12},\\ -3(x-24)(x^2-36x+216) & \text{if } p\equiv 11\pmod{12}.\end{cases}$$

Taking $x=54/(t+1)$ in (3.5) and applying the above we obtain

(3.6)

$$\begin{aligned}&W_{[\frac p3]}((t-1)/(t+1))\\&\equiv\begin{cases}(\frac{27(1-t)}{1+t})^{[\frac p{12}]}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\pmod p & \text{if } p\equiv 1\pmod{12},\\ \frac{18(4t-5)}{t+1}(\frac{27(1-t)}{1+t})^{[\frac p{12}]}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\pmod p & \text{if } p\equiv 5\pmod{12},\\ \frac{108(2t^2-14t+11)}{(t+1)^2}(\frac{27(1-t)}{1+t})^{[\frac p{12}]}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\pmod p & \text{if } p\equiv 7\pmod{12},\\ \frac{1944(4t-5)(2t^2-14t+11)}{(t+1)^3}(\frac{27(1-t)}{1+t})^{[\frac p{12}]}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\pmod p & \text{if } p\equiv 11\pmod{12}.\end{cases}\end{aligned}$$

Taking $x=(t-1)/(t+1)$ in (3.2) we get

$$P_{[\frac p3]}(t)=\Big(\frac{t+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big). \tag{3.7}$$

If $p\equiv 2\pmod 3$ and $t\equiv\frac54\pmod p$, from the above we get

$$P_{[\frac p3]}(t)\equiv P_{[\frac p3]}\Big(\frac54\Big)=\Big(\frac{\frac54+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{\frac54-1}{\frac54+1}\Big)\equiv 0\pmod p.$$

On the other hand,

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{x^3-27/4}{p}\Big)=\sum_{y=0}^{p-1}\Big(\frac{y-27/4}{p}\Big)=0.$$

Thus the result is true when $p \equiv 2 \pmod 3$ and $t \equiv \frac{5}{4} \pmod p$. If $p \equiv 3 \pmod 4$ and $2t^2 - 14t + 11 \equiv 0 \pmod p$, from (3.6) and (3.7) we deduce that

$$P_{[\frac{p}{3}]}(t) = \Big(\frac{t+1}{2}\Big)^{[\frac{p}{3}]} W_{[\frac{p}{3}]}\Big(\frac{t-1}{t+1}\Big) \equiv 0 \pmod p.$$

As

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x}{p}\Big) = \sum_{x=0}^{p-1}\Big(\frac{(-x)^3+3(4t-5)(-x)}{p}\Big) = -\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x}{p}\Big),$$

we see that

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big) = \sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x}{p}\Big) = 0.$$

Thus the result is true when $p \equiv 3 \pmod 4$ and $2t^2 - 14t + 11 \equiv 0 \pmod p$. Set $m = 3(4t-5)$ and $n = 2(2t^2 - 14t + 11)$. Then

$$\text{(3.8)} \qquad 4m^3 + 27n^2 = -432(1-t)(1+t)^3 \quad \text{and so} \quad \frac{2^8\cdot 3^3 m^3}{4m^3+27n^2} = \frac{432(5-4t)^3}{(1-t)(1+t)^3}.$$

By the above, we may assume that $m \not\equiv 0 \pmod p$ for $p \equiv 2 \pmod 3$ and $n \not\equiv 0 \pmod p$ for $p \equiv 3 \pmod 4$. From (3.3) we see that

(3.9)

$$\begin{aligned} &J_p\Big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\Big) = J_p\Big(\frac{2^8\cdot 3^3m^3}{4m^3+27n^2}\Big) \\ &\equiv -(-48m)^{\frac{(\frac{p}{3})-1}{2}}(864n)^{\frac{(\frac{-1}{p})-1}{2}}(-16(4m^3+27n^2))^{-[\frac{p}{12}]}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big) \pmod p. \end{aligned}$$

If $p \equiv 1 \pmod{12}$, from (3.6)-(3.9) we deduce that

$$\begin{aligned} &P_{[\frac{p}{3}]}(t) \\ &= \Big(\frac{t+1}{2}\Big)^{[\frac{p}{3}]} W_{[\frac{p}{3}]}\Big(\frac{t-1}{t+1}\Big) \equiv \Big(\frac{1+t}{2}\Big)^{\frac{p-1}{3}}\Big(\frac{27(1-t)}{1+t}\Big)^{\frac{p-1}{12}} J_p\Big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\Big) \\ &\equiv -2^{-\frac{p-1}{3}}(3(1+t))^{\frac{p-1}{4}}(1-t)^{\frac{p-1}{12}}(2^8\cdot 3^3(1-t)(1+t)^3)^{-\frac{p-1}{12}}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big) \\ &\equiv -\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big) \pmod p. \end{aligned}$$

If $p\equiv 5\pmod{12}$, from (3.6)-(3.9) we deduce that

$$\begin{aligned}
&P_{[\frac p3]}(t)\\
&=\Big(\frac{t+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big)\equiv\Big(\frac{t+1}{2}\Big)^{\frac{p-2}{3}}\frac{18(4t-5)}{t+1}\Big(\frac{27(1-t)}{1+t}\Big)^{\frac{p-5}{12}}J_p\Big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\Big)\\
&\equiv 2^{-\frac{p-5}{3}}3^{\frac{p+3}{4}}(4t-5)(1+t)^{\frac{p-5}{4}}(1-t)^{\frac{p-5}{12}}(144(4t-5))^{-1}\\
&\qquad\times(2^8\cdot3^3(1-t)(1+t)^3)^{-\frac{p-5}{12}}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\
&\equiv\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\pmod p.
\end{aligned}$$

If $p\equiv 7\pmod{12}$, from (3.6)-(3.9) we deduce that

$$\begin{aligned}
P_{[\frac p3]}(t)&=\Big(\frac{t+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big)\\
&\equiv\Big(\frac{t+1}{2}\Big)^{\frac{p-1}{3}}\frac{108(2t^2-14t+11)}{(t+1)^2}\Big(\frac{27(1-t)}{1+t}\Big)^{\frac{p-7}{12}}J_p\Big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\Big)\\
&\equiv-2^{-\frac{p-7}{3}}3^{\frac{p+5}{4}}(2t^2-14t+11)(1+t)^{\frac{p-7}{4}}(1-t)^{\frac{p-7}{12}}(2^6\cdot3^3(2t^2-14t+11))^{-1}\\
&\qquad\times(2^8\cdot3^3(1-t)(1+t)^3)^{-\frac{p-7}{12}}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\
&\equiv-\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\pmod p.
\end{aligned}$$

If $p\equiv 11\pmod{12}$, from (3.6)-(3.9) we deduce that

$$\begin{aligned}
P_{[\frac p3]}(t)&=\Big(\frac{t+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big)\\
&\equiv\Big(\frac{t+1}{2}\Big)^{\frac{p-2}{3}}\frac{1944(4t-5)(2t^2-14t+11)}{(t+1)^3}\Big(\frac{27(1-t)}{1+t}\Big)^{\frac{p-11}{12}}J_p\Big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\Big)\\
&\equiv 2^{-\frac{p-11}{3}}3^{\frac{p-11}{4}+5}(4t-5)(2t^2-14t+11)(1+t)^{\frac{p-11}{4}}(1-t)^{\frac{p-11}{12}}(48m)^{-1}(864n)^{-1}\\
&\qquad\times(2^8\cdot3^3(1-t)(1+t)^3)^{-\frac{p-11}{12}}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\
&\equiv\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\pmod p.
\end{aligned}$$

This completes the proof of the theorem.

Corollary 3.1. *Let $p>3$ be a prime and let t be a variable. Then*

$$\begin{aligned}&\sum_{k=0}^{[p/3]}\binom{2k}{k}\binom{3k}{k}\Big(\frac{1-t}{54}\Big)^k\\&\equiv P_{[\frac p3]}(t)\equiv-\Big(\frac p3\Big)\sum_{x=0}^{p-1}(x^3+3(4t-5)x+2(2t^2-14t+11))^{\frac{p-1}{2}} \pmod p.\end{aligned}$$

Proof. From [S2, Lemma 2.3] we have $P_{[\frac p3]}(t)\equiv\sum_{k=0}^{[p/3]}\binom{2k}{k}\binom{3k}{k}(\frac{1-t}{54})^k \pmod p$. By Theorem 3.1 and Euler's criterion, the result is true for $t=0,1,\ldots,p-1$. Since both sides are polynomials in t with degree at most $p-1$, using Lagrange's theorem we obtain the result.

Corollary 3.2. *Let $p>3$ be a prime and $t\in R_p$. Then*

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\\&=\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(4t+5)x+2(2t^2+14t+11)}{p}\Big).\end{aligned}$$

Proof. Since $P_{[\frac p3]}(-t)=(-1)^{[\frac p3]}P_{[\frac p3]}(t)=(\frac p3)P_{[\frac p3]}(t)$, by Theorem 3.1 we have

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\\&\equiv\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(4t+5)x+2(2t^2+14t+11)}{p}\Big) \pmod p.\end{aligned}$$

By Hasse's estimate ([C, Theorem 14.12, p.315]),

$$\Big|\sum_{x=0}^{p-1}\Big(\frac{x^3-3(\pm4t+5)x+2(2t^2\pm14t+11)}{p}\Big)\Big|\le2\sqrt p.$$

For $p\ge17$ we have $4\sqrt p<p$, from the above we deduce the result. For $p\in\{5,11,13\}$ and $t\in\{0,1,\ldots,p-1\}$ one can easily verify that the result is also true. Thus the corollary is proved.

Corollary 3.3. *Let $p>3$ be a prime. Then*

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-120x+506}{p}\Big)=\begin{cases}(\frac 2p)L & \text{if } 3\mid p-1,\ 4p=L^2+27M^2 \text{ and } 3\mid L-1,\\ 0 & \text{if } p\equiv2\pmod 3.\end{cases}$$

Proof. Taking $t=\frac54$ in Corollary 3.2 we find that

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{27}{4}}{p}\Big)=\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-30x+\frac{253}{4}}{p}\Big)=\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{(\frac x2)^3-30\cdot\frac x2+\frac{253}{4}}{p}\Big)$$
$$=\Big(\frac p3\Big)\Big(\frac 2p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-120x+506}{p}\Big).$$

For $p\equiv 2\pmod 3$ it is clear that $\sum_{x=0}^{p-1}\big(\frac{x^3-\frac{27}{4}}{p}\big)=\sum_{x=0}^{p-1}\big(\frac{x-\frac{27}{4}}{p}\big)=\sum_{x=0}^{p-1}\big(\frac xp\big)=0$. Thus the result is true when $p\equiv 2\pmod 3$.

Now assume that $p\equiv 1\pmod 3$, $p=A^2+3B^2$, $4p=L^2+27M^2$ and $A\equiv L\equiv 1\pmod 3$. It is known that $2^{\frac{p-1}{3}}\equiv 1\pmod p$ if and only if $3\mid B$. When $3\nmid B$ we choose the sign of B so that $B\equiv 1\pmod 3$. By [S1, (2.12)], $2^{(p-1)/3}\equiv\frac12(-1-\frac AB)\pmod p$. From [S1, (2.9)-(2.11)] we deduce that

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-27/4}{p}\Big)=\begin{cases}-2A=L & \text{if } 2^{\frac{p-1}{3}}\equiv 1\pmod p,\\ A+3B=L & \text{if } 2^{\frac{p-1}{3}}\not\equiv 1\pmod p \text{ and } B\equiv 1\pmod 3.\end{cases}$$

Thus

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-120x+506}{p}\Big)=\Big(\frac 2p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-27/4}{p}\Big)=\Big(\frac 2p\Big)L.$$

This completes the proof.

Theorem 3.2. *Let $p>3$ be a prime.*

(i) *If $p\equiv 2\pmod 3$, then*

$$\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\equiv\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv P_{[\frac p3]}\Big(\frac54\Big)\equiv 0\pmod p.$$

(ii) *If $p\equiv 1\pmod 3$ and so $4p=L^2+27M^2$ with $L,M\in\mathbb Z$ and $L\equiv 1\pmod 3$, then*

$$\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\equiv\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv P_{[\frac p3]}\Big(\frac54\Big)\equiv -L\equiv\Big(\frac{-2}{p}\Big)\binom{\frac{2(p-1)}{3}}{[\frac{p}{12}]}\pmod p.$$

Proof. Putting $t=\pm\frac54$ in Corollary 3.1 we get

$$P_{[\frac p3]}\Big(\frac54\Big)\equiv\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\pmod p\quad\text{and}\quad P_{[\frac p3]}\Big(-\frac54\Big)\equiv\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\pmod p.$$

This together with (1.3) yields

$$\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\equiv\Big(\frac p3\Big)\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\pmod p.$$

If $p\equiv 2\pmod 3$, from the proof of Theorem 3.1 we know that $P_{[\frac p3]}(\frac 54)\equiv 0\pmod p$. Thus (i) is true.

Now assume that $p\equiv 1\pmod 3$, $4p=L^2+27M^2$ and $L\equiv 1\pmod 3$. By Theorem 3.1 and the proof of Corollary 3.3 we have

$$P_{[\frac p3]}\Big(\frac 54\Big)\equiv -\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{27}{4}}{p}\Big)=-L\pmod p.$$

On the other hand, by the proof of Theorem 3.1,

$$\begin{aligned}P_{[\frac p3]}\Big(\frac 54\Big)&=\Big(\frac{\frac 54+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{\frac 54-1}{\frac 54+1}\Big)\\ &\equiv\begin{cases}\big(\frac 98\big)^{\frac{p-1}{3}}\big(\frac{27(1-\frac 54)}{1+\frac 54}\big)^{\frac{p-1}{12}}J_p(0)\equiv(-1)^{\frac{p-1}{12}}3^{-\frac{p-1}{4}}J_p(0)\pmod p & \text{if } p\equiv 1\pmod{12},\\ \big(\frac 98\big)^{\frac{p-1}{3}}\frac{108(2(\frac 54)^2-14\cdot\frac 54+11)}{(\frac 54+1)^2}\big(\frac{27(1-\frac 54)}{1+\frac 54}\big)^{\frac{p-7}{12}}J_p(0)\\ \quad\equiv -8(-1)^{\frac{p-7}{12}}3^{-\frac{p-7}{4}}J_p(0)\pmod p & \text{if } p\equiv 7\pmod{12}.\end{cases}\end{aligned}$$

By (3.4),

$$\begin{aligned}J_p(0)&=1728^{[\frac p{12}]}P_{[\frac p{12}]}^{(-\frac 13,-\frac 12(\frac{-1}{p}))}(1)\\ &=1728^{[\frac p{12}]}\cdot 2^{-[\frac p{12}]}\sum_{r=0}^{[\frac p{12}]}\binom{[\frac p{12}]-\frac 13}{r}\binom{[\frac p{12}]-\frac 12(\frac{-1}{p})}{[\frac p{12}]-r}0^{[\frac p{12}]-r}2^r\\ &=1728^{[\frac p{12}]}\binom{[\frac p{12}]-\frac 13}{[\frac p{12}]}=(-1728)^{[\frac p{12}]}\binom{\frac 13-1}{[\frac p{12}]}\\ &\equiv(-1728)^{[\frac p{12}]}\binom{\frac{2(p-1)}{3}}{[\frac p{12}]}\pmod p.\end{aligned}$$

Therefore

$$P_{[\frac p3]}\Big(\frac 54\Big)\equiv\begin{cases}(-1)^{\frac{p-1}{12}}3^{-\frac{p-1}{4}}(-1728)^{\frac{p-1}{12}}\binom{\frac{2(p-1)}{3}}{\frac{p-1}{12}}\equiv\big(\frac 2p\big)\binom{\frac{2(p-1)}{3}}{\frac{p-1}{12}}\pmod p & \text{if } 12\mid p-1,\\ -8(-1)^{\frac{p-7}{12}}3^{-\frac{p-7}{4}}(-1728)^{\frac{p-7}{12}}\binom{\frac{2(p-1)}{3}}{\frac{p-7}{12}}\equiv-\big(\frac 2p\big)\binom{\frac{2(p-1)}{3}}{\frac{p-7}{12}}\pmod p & \text{if } 12\mid p-7.\end{cases}$$

Now putting all the above together we obtain the result.

Remark 3.1 For any prime $p>3$, Zhi-Wei Sun conjectured that ([Su1, Conjecture A46])

$$\sum_{k=0}^{p-1}\frac{(3k)!}{24^k\cdot k!^3}\equiv\Big(\frac p3\Big)\sum_{k=0}^{p-1}\frac{(3k)!}{(-216)^k\cdot k!^3}\equiv\begin{cases}\binom{2(p-1)/3}{(p-1)/3}\pmod{p^2} & \text{if } p\equiv 1\pmod 3,\\ \binom{2(p+1)/3}{(p+1)/3}^{-1}p\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

4. Congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/m^k$.

Let $p>3$ be a prime and $m\in\mathbb{Z}$ with $p\nmid m$. In the section we partially solve Z.W. Sun's conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/m^k \pmod{p^2}$.

Theorem 4.1. *Let $p>3$ be a prime, $m\in R_p$, $m\not\equiv 0 \pmod p$ and $t=\sqrt{1-108/m}$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}\equiv P_{[\frac p3]}(t)^2\equiv\Big(\sum_{x=0}^{p-1}(x^3+3(4t-5)x+2(2t^2-14t+11))^{\frac{p-1}{2}}\Big)^2 \pmod p.$$

Moreover, if $P_{[\frac p3]}(t)\equiv 0 \pmod p$ or $\sum_{x=0}^{p-1}(x^3+3(4t-5)x+2(2t^2-14t+11))^{\frac{p-1}{2}}\equiv 0 \pmod p$, then $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/m^k\equiv 0 \pmod{p^2}$.

Proof. Since $\frac{1-t}{54}(1-27\cdot\frac{1-t}{54})=\frac1m$, by Theorem 2.1 we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}\equiv\Big(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\Big(\frac{1-t}{54}\Big)^k\Big)^2 \pmod{p^2}. \tag{4.1}$$

Observe that $p\mid\binom{2k}{k}\binom{3k}{k}$ for $[\frac p3]<k<p$. From Corollary 3.1 we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\Big(\frac{1-t}{54}\Big)^k\\&\equiv P_{[\frac p3]}(t)\equiv-\Big(\frac p3\Big)\sum_{x=0}^{p-1}(x^3+3(4t-5)x+2(2t^2-14t+11))^{\frac{p-1}{2}} \pmod p.\end{aligned}$$

This together with (4.1) yields the result.

Theorem 4.2 ([Su1, Conjecture A8]). *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv\begin{cases}L^2 \pmod p & \text{if } p\equiv 1 \pmod 3 \text{ and so } 4p=L^2+27M^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Proof. Putting $m=-192$ and $t=\frac54$ in Theorem 4.1 and then applying Theorem 3.2 we obtain the result.

Lemma 4.1. *Let p be an odd prime and let a,m,n be algebraic numbers which are integral for p. Then*

$$\sum_{x=0}^{p-1}(x^3+a^2mx+a^3n)^{\frac{p-1}{2}}\equiv a^{\frac{p-1}{2}}\sum_{x=0}^{p-1}(x^3+mx+n)^{\frac{p-1}{2}} \pmod p.$$

Moreover, if a,m,n are congruent to rational integers modulo p, then

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+a^2mx+a^3n}{p}\Big)=\Big(\frac ap\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big).$$

Proof. For any positive integer k it is well known that (see [IR, Lemma 2, p.235])

$$\sum_{x=0}^{p-1} x^k \equiv \begin{cases} p-1 \pmod p & \text{if } p-1 \mid k, \\ 0 \pmod p & \text{if } p-1 \nmid k. \end{cases}$$

Since

$$\begin{aligned}
&\sum_{x=0}^{p-1}(x^3+a^2mx+a^3n)^{\frac{p-1}{2}} \\
&= \sum_{x=0}^{p-1}\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}(x^3+a^2mx)^k(a^3n)^{\frac{p-1}{2}-k} \\
&= \sum_{x=0}^{p-1}\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}\sum_{r=0}^{k}\binom{k}{r}x^{3r}(a^2mx)^{k-r}(a^3n)^{\frac{p-1}{2}-k} \\
&= \sum_{r=0}^{(p-1)/2}\sum_{k=r}^{(p-1)/2}\binom{(p-1)/2}{k}\binom{k}{r}(a^2m)^{k-r}(a^3n)^{\frac{p-1}{2}-k}\sum_{x=0}^{p-1}x^{k+2r} \\
&\equiv (p-1)\sum_{r=0}^{(p-1)/2}\binom{(p-1)/2}{p-1-2r}\binom{p-1-2r}{r}(a^2m)^{p-1-3r}(a^3n)^{2r-\frac{p-1}{2}} \\
&= a^{\frac{p-1}{2}}(p-1)\sum_{\frac{p-1}{4}\le r\le \frac{p-1}{3}}\binom{(p-1)/2}{p-1-2r}\binom{p-1-2r}{r}m^{p-1-3r}n^{2r-\frac{p-1}{2}} \pmod p,
\end{aligned}$$

we see that the congruence in Lemma 4.1 is true.

Now suppose that a, m, n are congruent to rational integers modulo p. If $a \equiv 0 \pmod p$, then

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+a^2mx+a^3n}{p}\Big) = \sum_{x=0}^{p-1}\Big(\frac{x^3}{p}\Big) = \sum_{x=0}^{p-1}\Big(\frac{x}{p}\Big) = 0 = \Big(\frac{a}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big).$$

If $a \not\equiv 0 \pmod p$, then

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+a^2mx+a^3n}{p}\Big) = \sum_{x=0}^{p-1}\Big(\frac{(ax)^3+a^2m(ax)+a^3n}{p}\Big) = \Big(\frac{a}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big).$$

Thus the lemma is proved.

Lemma 4.2. *Let p be an odd prime. Then*

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-30x-56}{p}\Big) = \begin{cases} (-1)^{[\frac{p}{8}]+1}(\frac{3}{p})2c & \textit{if } p\equiv 1,3 \pmod 8,\ p=c^2+2d^2 \textit{ and } 4\mid c-1, \\ 0 & \textit{if } p\equiv 5,7 \pmod 8. \end{cases}$$

Proof. From [BE, Theorems 5.12 and 5.17] we know that

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-4x^2+2x}{p}\Big)=\begin{cases}(-1)^{[\frac p8]+1}2c & \text{if } p=c^2+2d^2\equiv 1,3 \pmod 8 \text{ with } 4\mid c-1,\\ 0 & \text{if } p\equiv 5,7 \pmod 8).\end{cases}$$

As $27(x^3-4x^2+2x)=(3x-4)^3-30(3x-4)-56$, we see that

$$\Big(\frac 3p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-4x^2+2x}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{x^3-30x-56}{p}\Big).$$

Thus the result follows.

Lemma 4.3. *Let p be an odd prime. Then*

$$\begin{aligned}&\sum_{n=0}^{p-1}(n^3-(15+30\sqrt{-2})n-28+70\sqrt{-2}))^{\frac{p-1}2}\\ &\equiv\begin{cases}(\frac{2+\sqrt{-2}}{p})(-1)^{[\frac p8]+1}(\frac 3p)2c \pmod p & \text{if } p=c^2+2d^2\equiv 1,3 \pmod 8 \text{ and } 4\mid c-1,\\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8).\end{cases}\end{aligned}$$

Proof. It is easily seen that

$$-15(1+2\sqrt{-2})=-30\Big(\frac{1-\sqrt{-2}}{\sqrt{-2}}\Big)^2\quad\text{and}\quad -28+70\sqrt{-2}=-56\Big(\frac{1-\sqrt{-2}}{\sqrt{-2}}\Big)^3.$$

Thus, by Lemmas 4.1 and 4.2 we have

$$\begin{aligned}&\sum_{n=0}^{p-1}\big(n^3-(15+30\sqrt{-2})n-28+70\sqrt{-2})\big)^{\frac{p-1}2}\\ &\equiv\Big(\frac{1-\sqrt{-2}}{\sqrt{-2}}\Big)^{\frac{p-1}2}\sum_{n=0}^{p-1}(n^3-30n-56)^{\frac{p-1}2}\equiv\Big(\frac{2+\sqrt{-2}}{-2}\Big)^{\frac{p-1}2}\sum_{n=0}^{p-1}\Big(\frac{n^3-30n-56}{p}\Big)\\ &\equiv\begin{cases}(\frac{2+\sqrt{-2}}{p})(-1)^{[\frac p8]+1}(\frac 3p)2c \pmod p & \text{if } p=c^2+2d^2\equiv 1,3 \pmod 8 \text{ and } 4\mid c-1,\\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8).\end{cases}\end{aligned}$$

This proves the lemma.

Theorem 4.3. *Let $p>3$ be a prime. Then*

$$\begin{aligned}&P_{[\frac p3]}(5/\sqrt{-2})\\ &\equiv\begin{cases}(-1)^{[\frac p8]}(\frac{-2-\sqrt{-2}}{p})2c \pmod p & \text{if } p=c^2+2d^2\equiv 1,3 \pmod 8 \text{ and } 4\mid c-1,\\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8)\end{cases}\end{aligned}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k}\equiv\begin{cases}4c^2 \pmod p & \text{if } p=c^2+2d^2\equiv 1,3 \pmod 8),\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8).\end{cases}$$

Proof. From Corollary 3.1 (with $t = 5/\sqrt{-2}$), Lemma 4.3 and Theorem 4.1 (with $m = 8$ and $t = 5/\sqrt{-2}$) we deduce the result.

Remark 4.1 Let p be an odd prime. Zhi-Wei Sun conjectured that ([Su1, Conjecture A5])

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k}\equiv\begin{cases}4c^2-2p \pmod{p^2} & \text{if } p=c^2+2d^2\equiv 1,3 \pmod 8,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Lemma 4.4. *Let p be an odd prime with $p \neq 11$. Then*

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-24\cdot 11x+14\cdot 11^2}{p}\Big)=\begin{cases}(\frac{6}{p})(\frac{u}{11})u & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=u^2+11v^2,\\ 0 & \text{if } (\frac{p}{11})=-1.\end{cases}$$

Proof. It is known that (see [RP] and [JM])

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-96\cdot 11x+112\cdot 11^2}{p}\Big)=\begin{cases}(\frac{3}{p})(\frac{u}{11})u & \text{if } (\frac{p}{11})=1 \text{ and } 4p=u^2+11v^2,\\ 0 & \text{if } (\frac{p}{11})=-1.\end{cases}$$

Since $(2x)^3-96\cdot 11\cdot 2x+112\cdot 11^2=8(x^3-24\cdot 11x+14\cdot 11^2)$, we deduce the result.

Lemma 4.5. *Let $p \neq 11$ be an odd prime. Then*

$$\begin{aligned}&\sum_{n=0}^{p-1}\Big(n^3+3(-5+\sqrt{-11})n+\frac{7}{4}(11-4\sqrt{-11})\Big)^{\frac{p-1}{2}}\\ &\equiv\begin{cases}(\frac{-11+\sqrt{-11}}{p})(\frac{6}{p})(\frac{u}{11})u \pmod p & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=u^2+11v^2,\\ 0 \pmod p & \text{if } (\frac{p}{11})=-1.\end{cases}\end{aligned}$$

Proof. It is easily seen that

$$3(-5+\sqrt{-11})=-24\cdot 11\Big(\frac{\sqrt{-11}+1}{4\sqrt{-11}}\Big)^2 \text{ and } \frac{7}{4}(11-4\sqrt{-11})=14\cdot 11^2\Big(\frac{\sqrt{-11}+1}{4\sqrt{-11}}\Big)^3.$$

Thus, by Lemma 4.1 we have

$$\begin{aligned}&\sum_{n=0}^{p-1}\Big(n^3+3(-5+\sqrt{-11})n+\frac{7}{4}(11-4\sqrt{-11})\Big)^{\frac{p-1}{2}}\\ &\equiv\Big(\frac{\sqrt{-11}+1}{4\sqrt{-11}}\Big)^{\frac{p-1}{2}}\sum_{x=0}^{p-1}\big(x^3-24\cdot 11x+14\cdot 11^2\big)^{\frac{p-1}{2}}\\ &\equiv\Big(\frac{-11+\sqrt{-11}}{-11}\Big)^{\frac{p-1}{2}}\sum_{x=0}^{p-1}\Big(\frac{x^3-24\cdot 11x+14\cdot 11^2}{p}\Big) \pmod p.\end{aligned}$$

Now applying Lemma 4.4 we deduce the result.

Theorem 4.4. *Let $p\neq 11$ be an odd prime. Then*

$$P_{[\frac p3]}\Big(\frac{\sqrt{-11}}{4}\Big)\equiv\begin{cases}-(\frac{-2}{p})(\frac{-11+\sqrt{-11}}{p})(\frac{u}{11})u\pmod p & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=u^2+11v^2,\\ 0\pmod p & \text{if } (\frac{p}{11})=-1\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv\begin{cases}u^2\pmod p & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=u^2+11v^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{11})=-1.\end{cases}$$

Proof. From Corollary 3.1, Lemma 4.5 and Theorem 4.1 (with $m=64$ and $t=\frac{\sqrt{-11}}{4}$) we deduce the result.

Remark 4.2 Let p be an odd prime such that $p\neq 11$. Zhi-Wei Sun conjectured that ([Su2, Conjecture 5.4])

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv\begin{cases}u^2-2p\pmod{p^2} & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=u^2+11v^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{11})=-1.\end{cases}$$

Let $p>3$ be a prime and let $\mathbb{F}_p$ be the field of p elements. For $m,n\in\mathbb{F}_p$ let $\#E_p(x^3+mx+n)$ be the number of points on the curve E: $y^2=x^3+mx+n$ over the field $\mathbb{F}_p$. It is well known that (see for example [S1, pp.221-222])

$$\#E_p(x^3+mx+n)=p+1+\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big). \tag{4.2}$$

Let $K=\mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and the curve $y^2=x^3+mx+n$ has complex multiplication by an order in K. By Deuring's theorem ([C, Theorem 14.16],[PV],[I]), we have

$$\#E_p(x^3+mx+n)=\begin{cases}p+1 & \text{if } p \text{ is inert in } K,\\ p+1-\pi-\bar\pi & \text{if } p=\pi\bar\pi \text{ in } K,\end{cases} \tag{4.3}$$

where π is in an order in K and $\bar\pi$ is the conjugate number of π. If $4p=u^2+dv^2$ with $u,v\in\mathbb{Z}$, we may take $\pi=\frac12(u+v\sqrt{-d})$. Thus,

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)=\begin{cases}\pm u & \text{if } 4p=u^2+dv^2 \text{ with } u,v\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases} \tag{4.4}$$

In [Gr], [JM] and [PV] the sign of u in (4.4) was determined for those imaginary quadratic fields K with class number 1. In [LM] and [I] the sign of u in (4.4) was determined for imaginary quadratic fields K with class number 2. For general results on the sign of u in (4.4), see [M], [St], [RS] and the survey [Si].

Lemma 4.6. *Let p be a prime with $p \equiv \pm 1 \pmod 8$. Then*

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+(-15+6\sqrt{2})n+24-14\sqrt{2}}{p}\Big) = \begin{cases} 2x(\frac{2x}{3}) & \text{if } p \equiv 1,7 \pmod{24} \text{ and so } p = x^2+6y^2, \\ 0 & \text{if } p \equiv 17,23 \pmod{24}. \end{cases}$$

Proof. From [I, p.133] we know that the elliptic curve defined by the equation $y^2 = x^3+(-21+12\sqrt{2})x-28+22\sqrt{2}$ has complex multiplication by the order of discriminant -24. Since $4p = u^2+24v^2$ implies $2 \mid u$ and $p = (\frac{u}{2})^2+6v^2$, by (4.4) and [I, Theorem 3.1] we have

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+(-21+12\sqrt{2})n-28+22\sqrt{2}}{p}\Big) = \begin{cases} 2x(\frac{2x}{3})(\frac{1+\sqrt{2}}{p}) & \text{if } p \equiv 1,7 \pmod{24} \text{ and so } p = x^2+6y^2, \\ 0 & \text{if } p \equiv 17,23 \pmod{24}. \end{cases}$$

Observe that

$$\frac{-15-6\sqrt{2}}{-21+12\sqrt{2}} = (1+\sqrt{2})^2 \quad \text{and} \quad \frac{24+14\sqrt{2}}{-28+22\sqrt{2}} = (1+\sqrt{2})^3.$$

Using Corollary 3.2 (with $t = 1/\sqrt{2}$) and Lemma 4.1 we see that

$$\begin{aligned} &\Big(\frac{p}{3}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3+(-15+6\sqrt{2})n+24-14\sqrt{2}}{p}\Big) \\ &= \sum_{n=0}^{p-1}\Big(\frac{n^3-(15+6\sqrt{2})n+24+14\sqrt{2}}{p}\Big) \\ &= \Big(\frac{1+\sqrt{2}}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3+(-21+12\sqrt{2})n-28+22\sqrt{2}}{p}\Big). \end{aligned}$$

Now putting all the above together we obtain the result.

Theorem 4.5. *Let p be a prime such that $p \equiv 1,7 \pmod 8$. Then*

$$P_{[\frac{p}{3}]}\Big(\frac{\sqrt{2}}{2}\Big) \equiv \begin{cases} 2x(\frac{x}{3}) \pmod p & \text{if } p = x^2+6y^2 \equiv 1,7 \pmod{24}, \\ 0 \pmod p & \text{if } p \equiv 17,23 \pmod{24} \end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p = x^2+6y^2 \equiv 1,7 \pmod{24}, \\ 0 \pmod{p^2} & \text{if } p \equiv 17,23 \pmod{24}. \end{cases}$$

Proof. From Theorem 3.1, Lemma 4.6 and Theorem 4.1 (with $m = 216$ and $t = \sqrt{2}/2$) we deduce the result.

Remark 4.3 For any prime $p > 3$, Z.W. Sun conjectured that ([Su1, Conjecture A14])

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ 8x^2-2p \pmod{p^2} & \text{if } p=2x^2+3y^2\equiv 5,11 \pmod{24},\\ 0 \pmod{p^2} & \text{if } p\equiv 13,17,19,23 \pmod{24}.\end{cases}$$

Lemma 4.7. *Let p be a prime with $p \equiv \pm 1 \pmod 5$. Then*

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+(-15+12\sqrt{5})n+42-28\sqrt{5}}{p}\Big) = \begin{cases} 2x(\frac{2x}{3}) & \textit{if } p\equiv 1,4 \pmod{15} \textit{ and so } p=x^2+15y^2,\\ 0 & \textit{if } p\equiv 11,14 \pmod{15}.\end{cases}$$

Proof. From [I, Proposition 3.3] we know that the elliptic curve defined by the equation $y^2 = x^3 + (105 + 48\sqrt{5})x - 784 - 350\sqrt{5}$ has complex multiplication by the order of discriminant -15. Since $4p = u^2 + 60v^2$ implies $2 \mid u$ and $p = (\frac{u}{2})^2 + 15v^2$, by (4.4) and [I, Theorem 3.1] we have

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+(105+48\sqrt{5})n-784-350\sqrt{5}}{p}\Big) = \begin{cases} 2x(\frac{2x}{3})(\frac{(1+\sqrt{5})/2}{p}) & \text{if } p\equiv 1,4 \pmod{15} \text{ and so } p=x^2+15y^2,\\ 0 & \text{if } p\equiv 11,14 \pmod{15}.\end{cases}$$

Observe that

$$\frac{-15+12\sqrt{5}}{105+48\sqrt{5}} = (\sqrt{5}-2)^2 \quad \text{and} \quad \frac{42-28\sqrt{5}}{-784-350\sqrt{5}} = (\sqrt{5}-2)^3.$$

Using Lemma 4.1 we see that

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+(-15+12\sqrt{5})n+42-28\sqrt{5}}{p}\Big) = \Big(\frac{\sqrt{5}-2}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3+(105+48\sqrt{5})n-784-350\sqrt{5}}{p}\Big).$$

Note that $(\sqrt{5}-2)\frac{1+\sqrt{5}}{2} = (\frac{1-\sqrt{5}}{2})^2$. We then have $(\frac{\sqrt{5}-2}{p}) = (\frac{(1+\sqrt{5})/2}{p})$. Now putting all the above together we obtain the result.

Theorem 4.6. *Let p be a prime such that $p\equiv 1,4 \pmod 5$. Then*

$$P_{[\frac p3]}(\sqrt5)\equiv\begin{cases}2x(\frac x3)\pmod p & \text{if } p=x^2+15y^2\equiv 1,4\pmod{15},\\ 0\pmod p & \text{if } p\equiv 11,14\pmod{15}\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\equiv\begin{cases}4x^2\pmod p & \text{if } p=x^2+15y^2\equiv 1,4\pmod{15},\\ 0\pmod{p^2} & \text{if } p\equiv 11,14\pmod{15}.\end{cases}$$

Proof. From Theorem 3.1, Lemma 4.7 and Theorem 4.1 (with $m=-27$ and $t=\sqrt5$) we deduce the result.

Lemma 4.8. *Let p be a prime such that $p\equiv\pm1 \pmod 5$. Then*

$$\begin{aligned}&\sum_{n=0}^{p-1}\Big(\frac{20n^3+(-300+108\sqrt5)n+521-252\sqrt5}{p}\Big)\\&=\begin{cases}(\frac x3)x & \text{if } p\equiv 1,4\pmod{15} \text{ and so } 4p=x^2+75y^2,\\ 0 & \text{if } p\equiv 11,14\pmod{15}.\end{cases}\end{aligned}$$

Proof. From [I, p.134] we know that the elliptic curve defined by the equation $y^2=x^3+(-2160+408\sqrt5)x+42130-10472\sqrt5$ has complex multiplication by the order of discriminant -75. By (4.4) and [I, Theorem 3.1] we have

$$\begin{aligned}&\sum_{n=0}^{p-1}\Big(\frac{n^3+(-2160+408\sqrt5)n+42130-10472\sqrt5}{p}\Big)\\&=\begin{cases}(\frac{-25-13\sqrt5}{p})(\frac x3)x & \text{if } p\equiv 1,4\pmod{15} \text{ and so } 4p=x^2+75y^2,\\ 0 & \text{if } p\equiv 11,14\pmod{15}.\end{cases}\end{aligned}$$

Observe that

$$\frac{-2160+408\sqrt5}{-300+108\sqrt5}=\Big(-\frac{7+\sqrt5}{2}\Big)^2\quad\text{and}\quad\frac{42130-10472\sqrt5}{-2520+1042\sqrt5}=\Big(-\frac{7+\sqrt5}{2}\Big)^3.$$

Using Lemma 4.1 we see that

$$\begin{aligned}&\sum_{n=0}^{p-1}\Big(\frac{n^3+(-2160+408\sqrt5)n+42130-10472\sqrt5}{p}\Big)\\&=\Big(\frac{-(7+\sqrt5)/2}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3+(-300+108\sqrt5)n-2520+1042\sqrt5}{p}\Big).\end{aligned}$$

Since $-\frac{7+\sqrt{5}}{2}(-25-13\sqrt{5})=2\sqrt{5}(3+2\sqrt{5})^2$ and

$$\begin{aligned}&\sum_{n=0}^{p-1}\Big(\frac{n^3+(-300+108\sqrt{5})n-2520+1042\sqrt{5}}{p}\Big)\\&=\sum_{n=0}^{p-1}\Big(\frac{(2\sqrt{5}n)^3+(-300+108\sqrt{5})(2\sqrt{5}n)-2520+1042\sqrt{5}}{p}\Big)\\&=\Big(\frac{2\sqrt{5}}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{20n^3+(-300+108\sqrt{5})n+521-252\sqrt{5}}{p}\Big),\end{aligned}$$

from the above we deduce the result.

Theorem 4.7. *Let p be a prime such that $p\equiv 1,4 \pmod 5$. Then*

$$P_{[\frac{p}{3}]}\Big(\frac{9}{20}\sqrt{5}\Big)\equiv\begin{cases}-(\frac{x}{3})x \pmod p & \text{if } p\equiv 1,4 \pmod{15} \text{ and so } 4p=x^2+75y^2,\\ 0 \pmod p & \text{if } p\equiv 11,14 \pmod{15}\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k}\equiv\begin{cases}x^2 \pmod p & \text{if } p\equiv 1,4 \pmod{15} \text{ and so } 4p=x^2+75y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 11,14 \pmod{15}.\end{cases}$$

Proof. From Theorem 3.1, Lemma 4.8 and Theorem 4.1 (with $m=-8640$ and $t=\frac{9}{4\sqrt{5}}$) we deduce the result.

Remark 4.4 Let $p>5$ be a prime. In [S2] the author made a conjecture equivalent to

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k}\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15} \text{ and so } 4p=x^2+75y^2,\\ 2p-3x^2 \pmod{p^2} & \text{if } p\equiv 7,13 \pmod{15} \text{ and so } 4p=3x^2+25y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Let $b\in\{17,41,89\}$ and $f(b)=-12^3,-48^3,-300^3$ according as $b=17,41,89$. In [Su1, Conjectures A20, A22 and A23], Z.W. Sun conjectured that for any odd prime $p\neq 3,b$,
(4.5)
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{f(b)^k}\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{b})=1 \text{ and so } 4p=x^2+3by^2,\\ 2p-3x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{b})=-1 \text{ and so } 4p=3x^2+by^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{3})=-(\frac{p}{b}).\end{cases}$$

Now we partially solve (4.5).

Theorem 4.8. *Let p be an odd prime such that $(\frac{17}{p})=1$. Then*

$$P_{[\frac{p}{3}]}\Big(\frac{\sqrt{17}}{4}\Big)\equiv\begin{cases}-(\frac{x}{3})x \pmod p & \text{if } p\equiv 1 \pmod 3 \text{ and so } 4p=x^2+51y^2,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}}\equiv\begin{cases}x^2\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=x^2+51y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3).\end{cases}$$

Proof. From [I, p.134] we know that the elliptic curve defined by the equation $y^2 = x^3-(60+12\sqrt{17})x-210-56\sqrt{17}$ has complex multiplication by the order of discriminant -51. Thus, by (4.4) and [I, Theorem 3.1] we have

$$\sum_{n=0}^{p-1}\Big(\frac{n^3-(60+12\sqrt{17})n-210-56\sqrt{17}}{p}\Big)$$
$$=\begin{cases}(\frac{-2}{p})(\frac{x}{3})x & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=x^2+51y^2,\\ 0 & \text{if } p\equiv 2\pmod 3).\end{cases}$$

It then follows from (1.3) and Theorem 3.1 that

$$P_{[\frac p3]}\Big(\frac{\sqrt{17}}{4}\Big)=\Big(\frac p3\Big)P_{[\frac p3]}\Big(-\frac{\sqrt{17}}{4}\Big)\equiv-\sum_{n=0}^{p-1}\Big(\frac{n^3+3(-\sqrt{17}-5)n+\frac{17}{4}+22+7\sqrt{17}}{p}\Big)$$
$$=-\sum_{n=0}^{p-1}\Big(\frac{(-\frac n2)^3-3(5+\sqrt{17})(-\frac n2)+\frac{105+28\sqrt{17}}{4}}{p}\Big)$$
$$=-\Big(\frac{-2}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-(60+12\sqrt{17})n-210-56\sqrt{17}}{p}\Big)$$
$$\equiv\begin{cases}-(\frac{x}{3})x\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=x^2+51y^2,\\ 0\pmod p & \text{if } p\equiv 2\pmod 3).\end{cases}$$

Taking $m=-12^3$ and $t=\frac{\sqrt{17}}{4}$ in Theorem 4.1 and then applying the above we deduce the remaining result.

Using [I, pp.134-135] and the method in the proof of Theorem 4.8 one can similarly prove Theorems 4.9 and 4.10.

Theorem 4.9. *Let p be an odd prime such that $(\frac{41}{p})=1$. Then*

$$P_{[\frac p3]}\Big(\frac{5\sqrt{41}}{32}\Big)\equiv\begin{cases}-(\frac{x}{3})x\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=x^2+123y^2,\\ 0\pmod p & \text{if } p\equiv 2\pmod 3).\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}}\equiv\begin{cases}x^2\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=x^2+123y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3).\end{cases}$$

Theorem 4.10. *Let $p > 5$ be a prime such that $(\frac{89}{p}) = 1$. Then*

$$P_{[\frac{p}{3}]}\Big(\frac{53\sqrt{89}}{500}\Big) \equiv \begin{cases} -(\frac{x}{3})x \pmod p & \text{if } p \equiv 1 \pmod 3 \text{ and so } 4p = x^2 + 267y^2, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}} \equiv \begin{cases} x^2 \pmod p & \text{if } p \equiv 1 \pmod 3 \text{ and so } 4p = x^2 + 267y^2, \\ 0 \pmod {p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

To conclude, we pose the following conjectures.

Conjecture 4.1. *For any prime $p > 5$ we have*

$$\sum_{k=0}^{p-1} \frac{9k+1}{(-8640)^k}\binom{2k}{k}^2\binom{3k}{k} \equiv p\Big(\frac{p}{15}\Big) \pmod{p^3},$$
$$\sum_{k=0}^{p-1} \frac{33k+4}{15^{3k}}\binom{2k}{k}^2\binom{3k}{k} \equiv 4p\Big(\frac{p}{3}\Big) \pmod{p^3},$$
$$\sum_{k=0}^{p-1} \frac{15k+2}{1458^k}\binom{2k}{k}^2\binom{3k}{k} \equiv (-1)^{\frac{p-1}{2}}2p \pmod{p^3}.$$

Conjecture 4.2. *Let p be a prime with $p \equiv \pm 1 \pmod 5$. Then*

$$\sum_{n=0}^{p-1} \Big(\frac{n^3 + 3(-125 + 44\sqrt{5})n + 154(21 - 10\sqrt{5})}{p}\Big)$$
$$= \begin{cases} 2x(\frac{2x}{3}) & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } p = x^2 + 15y^2, \\ 0 & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases}$$

Conjecture 4.3. *Let $p > 3$ be a prime. Then*

$$P_{[\frac{p}{3}]}\Big(\frac{5\sqrt{3}}{9}\Big) \equiv \begin{cases} (-1)^{\frac{p-1}{2}}2x(\frac{x}{3}) \pmod p & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Conjecture 4.4. *Let $p > 5$ be a prime. Then*

$$P_{[\frac{p}{3}]}\Big(\frac{11\sqrt{5}}{25}\Big) \equiv \begin{cases} 2x(\frac{x}{3}) \pmod p & \text{if } p \equiv 1, 19 \pmod{30} \text{ and so } p = x^2 + 15y^2, \\ -2y(\frac{y}{3})\sqrt{5} \pmod p & \text{if } p \equiv 17, 23 \pmod{30} \text{ and so } p = 3x^2 + 5y^2, \\ 0 \pmod p & \text{if } p \equiv 7, 11, 13, 29 \pmod{30}. \end{cases}$$

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