The paper will appear in Graphs and Combinatorics.

Some inversion formulas and formulas for Stirling numbers

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Abstract

In the paper we present some new inversion formulas and two new formulas for Stirling numbers.

MSC: 11B73,11A25,05A10,05A19,05A15
Keywords: Inversion formula; Inverse function; Stirling number

1. Introduction

Let $\mathbb{N}$ be the set of positive integers. Let $a(x) = x + a_2x^2 + a_3x^3 + \cdots$ and $\frac{a(x)^m}{m!} = \sum_{n=m}^{\infty} a(n,m)\frac{x^n}{n!}$ for $m \in \mathbb{N}$. In Section 2 we show that for any $k, n \in \mathbb{N}$,

$$a(n+k,n) = \sum_{r=1}^{k} \binom{k-n}{k-r}\binom{k+n}{k+r} a(k+r,r).$$

Let $f(x) = c_0 + c_1x + c_2x^2 + \cdots$ with $c_0 \neq 0$. In Section 3 we establish the following general inversion formula:

$$a_n = n\sum_{m=1}^{n} [x^{n-m}]f(x)^m \cdot b_m \quad (n = 1,2,3,\dots)$$
$$\iff b_n = \frac{1}{n}\sum_{m=1}^{n} [x^{n-m}]f(x)^{-n} \cdot a_m \quad (n = 1,2,3,\dots),$$

where $[x^k]g(x)$ is the coefficient of x^k in the power series expansion of $g(x)$. As a consequence, for a given complex number t we have the following inversion formula:

$$a_n = n\sum_{m=1}^{n} \binom{mt}{n-m} b_m \quad (n \ge 1) \iff b_n = \frac{1}{n}\sum_{m=1}^{n} \binom{-nt}{n-m} a_m \quad (n \ge 1).$$

Let $\alpha^{-1}(x)$ be the inverse function of $\alpha(x)$. In Section 4 we derive a general formula for $[x^{m+n}]\alpha(x)^m$ by using the power series expansion of $\alpha^{-1}(x)$. As a consequence, we deduce a symmetric inversion formula, see Theorem 4.3.

[1]The author was supported by the Natural Sciences Foundation of China (grant no. 10971078).

Suppose $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. Let $s(n, k)$ be the unsigned Stirling number of the first kind and $S(n, k)$ be the Stirling number of the second kind defined by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k$$

and

$$x^n = \sum_{k=0}^{n} S(n,k) x(x-1)\cdots(x-k+1).$$

In the paper we obtain new formulas for Stirling numbers, see Theorems 2.3 and 4.2.

2. The formula for $[x^m]f(x)^t$

Lemma 2.1. *Let t be a variable and $m \in \mathbb{N}$. Then*

$$\begin{aligned}
&[x^m](1 + a_1 x + a_2 x^2 + \cdots + a_m x^m + \cdots)^t \\
&= \sum_{k_1 + 2k_2 + \cdots + mk_m = m} \frac{t(t-1)\cdots(t-(k_1+\cdots+k_m)+1)}{k_1! \cdots k_m!} a_1^{k_1} \cdots a_m^{k_m}.
\end{aligned}$$

Proof. Using the binomial theorem and the multinomial theorem we see that

$$\begin{aligned}
&[x^m](1 + a_1 x + a_2 x^2 + \cdots + a_m x^m + \cdots)^t \\
&= [x^m](1 + a_1 x + a_2 x^2 + \cdots + a_m x^m)^t \\
&= [x^m] \sum_{n=0}^{\infty} \binom{t}{n} (a_1 x + a_2 x^2 + \cdots + a_m x^m)^n \\
&= \sum_{n=0}^{m} \binom{t}{n} [x^m](a_1 x + a_2 x^2 + \cdots + a_m x^m)^n \\
&= \sum_{n=0}^{m} \binom{t}{n} [x^m] \sum_{k_1 + k_2 + \cdots + k_m = n} \frac{n!}{k_1! \cdots k_m!} (a_1 x)^{k_1} \cdots (a_m x^m)^{k_m} \\
&= \sum_{n=0}^{m} \binom{t}{n} \sum_{\substack{k_1 + \cdots + k_m = n \\ k_1 + 2k_2 + \cdots + mk_m = m}} \frac{n!}{k_1! \cdots k_m!} a_1^{k_1} \cdots a_m^{k_m} \\
&= \sum_{k_1 + 2k_2 + \cdots + mk_m = m} \frac{t(t-1)\cdots(t-(k_1+\cdots+k_m)+1)}{k_1! \cdots k_m!} a_1^{k_1} \cdots a_m^{k_m}.
\end{aligned}$$

Theorem 2.1. *Let t be a variable, $m \in \mathbb{N}$ and $f(x) = 1 + a_1 x + a_2 x^2 + \cdots$. Then*

$$[x^m]f(x)^t = \sum_{r=1}^{m} \binom{m-t}{m-r} \binom{t}{r} [x^m] f(x)^r.$$

Proof. From Lemma 2.1 we see that $[x^m]f(x)^t$ is a polynomial of t with degree $\le m$. Hence

$$P_m(t) = [x^m]f(x)^t - \sum_{r=1}^{m} \binom{m-t}{m-r} \binom{t}{r} [x^m] f(x)^r$$

is also a polynomial of t with degree $\le m$. If $r \in \{1, 2, \ldots, m\}$ and $t \in \{0, 1, \ldots, m\}$ with $t \ne r$, then $t < r$ or $m - t < m - r$ and hence $\binom{m-t}{m-r}\binom{t}{r} = 0$. Thus $P_m(t) = 0$ for $t = 0, 1, \ldots, m$. Therefore $P_m(t) = 0$ for all t. This yields the result.

Corollary 2.1. *Let $m \in \mathbb{N}$ and let a be a complex number. Then*

$$\sum_{r=1}^{m} \binom{m+a}{m-r} (-1)^{m-r} \binom{a+r-1}{r} r^m = a^m.$$

Proof. Clearly $[x^m](e^x)^t = \frac{t^m}{m!}$. Thus, by Theorem 2.1 we have

$$\frac{t^m}{m!} = \sum_{r=1}^{m} \binom{m-t}{m-r} \binom{t}{r} \frac{r^m}{m!}.$$

Now taking $t = -a$ and noting that $\binom{-a}{r} = (-1)^r \binom{a+r-1}{r}$ we deduce the result.

Theorem 2.2. *Let $a(x) = x + a_2x^2 + a_3x^3 + \cdots$. For $m \in \mathbb{N}$ let $\frac{a(x)^m}{m!} = \sum_{n=m}^{\infty} a(n,m)\frac{x^n}{n!}$. Then for any $k, n \in \mathbb{N}$ we have*

$$a(n+k,n) = \sum_{r=1}^{k} \binom{k-n}{k-r} \binom{k+n}{k+r} a(k+r,r).$$

Proof. Set $\alpha(x) = a(x)/x$. Then for $m \in \mathbb{N}$ we have

$$\alpha(x)^m = \frac{a(x)^m}{x^m} = \sum_{k=0}^{\infty} a(m+k,m) \cdot \frac{m!}{(m+k)!} x^k.$$

Thus,

$$[x^k]\alpha(x)^n = a(n+k,n)\frac{n!}{(n+k)!} \quad \text{and} \quad [x^k]\alpha(x)^r = a(k+r,r)\frac{r!}{(k+r)!}.$$

Since $\alpha(0) = 1$, by Theorem 2.1 we have

$$[x^k]\alpha(x)^n = \sum_{r=1}^{k} \binom{k-n}{k-r} \binom{n}{r} [x^k]\alpha(x)^r.$$

Hence

$$a(n+k,n)\frac{n!}{(n+k)!} = \sum_{r=1}^{k} \binom{k-n}{k-r} \binom{n}{r} \frac{r!}{(k+r)!} a(k+r,r)$$

and so

$$a(n+k,n) = \sum_{r=1}^{k} \binom{k-n}{k-r} \frac{(n+k)!}{(n-r)!(k+r)!} a(k+r,r).$$

This is the result.

Theorem 2.3. *Let $k, n \in \mathbb{N}$. Then*

$$S(n+k,n) = \sum_{r=1}^{k} \binom{k-n}{k-r} \binom{k+n}{k+r} S(k+r,r)$$

and

$$s(n+k,n) = \sum_{r=1}^{k} \binom{k-n}{k-r} \binom{k+n}{k+r} s(k+r,r).$$

Proof. It is well known that ([5])

$$\frac{(e^x-1)^m}{m!} = \sum_{n=m}^{\infty} S(n,m)\frac{x^n}{n!} \quad \text{and} \quad \frac{(\log(1+x))^m}{m!} = \sum_{n=m}^{\infty} (-1)^{n-m} s(n,m)\frac{x^n}{n!}.$$

Thus the result follows from Theorem 2.2.

3. A general inversion formula involving $[x^k]f(x)^t$

Lemma 3.1. *Let $\alpha^{-1}(x)$ be the inverse function of $\alpha(x)$. Then for any two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ we have:*

$$a_n = \sum_{m=0}^{\infty}[x^n]\alpha(x)^m b_m \quad (n = 0, 1, 2, \ldots)$$
$$\iff b_n = \sum_{m=0}^{\infty}[x^n]\alpha^{-1}(x)^m a_m \quad (n = 0, 1, 2, \ldots).$$

Proof. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n$. Then clearly

$$a_n = \sum_{m=0}^{\infty}[x^n]\alpha(x)^m b_m \quad (n = 0, 1, 2, \ldots)$$
$$\iff a(x) = \sum_{m=0}^{\infty} b_m \sum_{n=0}^{\infty}[x^n]\alpha(x)^m x^n = \sum_{m=0}^{\infty} b_m \alpha(x)^m$$
$$\iff a(x) = b(\alpha(x)) \iff b(x) = a(\alpha^{-1}(x))$$
$$\iff b_n = \sum_{m=0}^{\infty}[x^n]\alpha^{-1}(x)^m a_m \quad (n = 0, 1, 2, \ldots).$$

So the lemma is proved.

Theorem 3.1. *Let $k \in \mathbb{N}$. For nonnegative integers m and n let*

$$\alpha_k(n, m) = \begin{cases} (-1)^{\frac{n}{k}}\binom{\frac{m}{k}}{\frac{n}{k}} & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

Then we have the following inversion formula:

$$a_n = \sum_{m=0}^{\infty}\alpha_k(n, m) b_m \quad (n = 0, 1, 2, \ldots)$$
$$\iff b_n = \sum_{m=0}^{\infty}\alpha_k(n, m) a_m \quad (n = 0, 1, 2, \ldots).$$

Proof. Let $\alpha(x) = (1 - x^k)^{\frac{1}{k}}$ $(0 < x < 1)$. Then clearly $\alpha^{-1}(x) = \alpha(x)$ and $\alpha(x)^m = (1-x^k)^{\frac{m}{k}} = \sum_{r=0}^{\infty}\binom{\frac{m}{k}}{r}(-1)^r x^{kr} = \sum_{n=0}^{\infty}\alpha_k(n, m)x^n$. Thus applying Lemma 3.1 we deduce the theorem.

Lemma 3.2 (Lagrange inversion formula ([5, p.148], [12, pp.36-44])). *Let $\alpha(x) = \alpha_1 x + \alpha_2 x^2 + \cdots$ with $\alpha_1 \neq 0$, and let $k, n \in \mathbb{N}$ with $k \leq n$. Then*

$$[x^n](\alpha^{-1}(x))^k = \frac{k}{n}[x^{n-k}]\Big(\frac{\alpha(x)}{x}\Big)^{-n}.$$

Theorem 3.2. *Let $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ with $c_0 \neq 0$. Then for any two sequences $\{a_n\}$ and $\{b_n\}$ we have the following inversion formula:*

$$a_n = n\sum_{m=1}^{n}[x^{n-m}]f(x)^m \cdot b_m \quad (n = 1, 2, 3, \ldots)$$
$$\iff b_n = \frac{1}{n}\sum_{m=1}^{n}[x^{n-m}]f(x)^{-n} \cdot a_m \quad (n = 1, 2, 3, \ldots).$$

Proof. Set $\alpha(x) = xf(x)$. Then clearly $[x^n]\alpha(x)^m = 0$ for $m > n$. As $\alpha^{-1}(xf(x)) = \alpha^{-1}(\alpha(x)) = x$ we see that $\alpha^{-1}(0) = 0$ and so $\alpha^{-1}(x) = d_1x + d_2x^2 + \cdots$ for some $d_1, d_2, \ldots$. Thus $[x^n]\alpha^{-1}(x)^m = 0$ for $m > n$. Set $a_0 = b_0 = 0$. From Lemma 3.1 we see that

$$a_n = \sum_{m=0}^{\infty}[x^n]\alpha(x)^m \cdot b_m = \sum_{m=1}^{n}[x^n]\alpha(x)^m \cdot b_m \quad (n \geq 1)$$
$$\iff b_n = \sum_{m=0}^{\infty}[x^n]\alpha^{-1}(x)^m \cdot a_m = \sum_{m=1}^{n}[x^n]\alpha^{-1}(x)^m \cdot a_m \quad (n \geq 1).$$

For $m \leq n$ we see that $[x^n]\alpha(x)^m = [x^n]x^m f(x)^m = [x^{n-m}]f(x)^m$ and $[x^n]\alpha^{-1}(x)^m = \frac{m}{n}[x^{n-m}]f(x)^{-n}$ by Lemma 3.2. Thus

$$a_n = \sum_{m=1}^{n}[x^{n-m}]f(x)^m \cdot b_m \ (n \geq 1) \iff b_n = \sum_{m=1}^{n}\frac{m}{n}[x^{n-m}]f(x)^{-n} \cdot a_m \ (n \geq 1).$$

Now substituting a_n by a_n/n we obtain the result.

As $\mathrm{e}^{cx} = \sum_{k=0}^{\infty}\frac{(cx)^k}{k!}$, we see that

$$[x^{n-m}](\mathrm{e}^x)^m = \frac{m^{n-m}}{(n-m)!} \quad \text{and} \quad [x^{n-m}](\mathrm{e}^x)^{-n} = \frac{(-n)^{n-m}}{(n-m)!}.$$

Thus, putting $f(x) = \mathrm{e}^x$ in Theorem 3.2 we have the following inversion formula:

$$a_n = n\sum_{m=1}^{n}\frac{m^{n-m}}{(n-m)!}b_m \quad (n \geq 1) \iff b_n = \frac{1}{n}\sum_{m=1}^{n}\frac{(-n)^{n-m}}{(n-m)!}a_m \quad (n \geq 1).$$

Substituting a_n by $a_n/(n-1)!$, and b_n by $b_n/n!$ we obtain

$$a_n = \sum_{m=1}^{n}\binom{n}{m}m^{n-m}b_m \quad (n \geq 1) \iff b_n = \sum_{m=1}^{n}\binom{n-1}{m-1}(-n)^{n-m}a_m \quad (n \geq 1).$$

This is a known result. See [11, p.96].

As $(1+x)^{ct} = \sum_{k=0}^{\infty}\binom{ct}{k}x^k$ $(|x| < 1)$, we see that for $1 \leq m \leq n$,

$$[x^{n-m}](1+x)^{mt} = \binom{mt}{n-m} \quad \text{and} \quad [x^{n-m}](1+x)^{-nt} = \binom{-nt}{n-m}.$$

Now putting $f(x) = (1+x)^t$ in Theorem 3.2 and applying the above we deduce the following result.

Theorem 3.3. *Let t be a complex number. For any two sequences $\{a_n\}$ and $\{b_n\}$ we have the following inversion formula:*

$$a_n = n\sum_{m=1}^{n}\binom{mt}{n-m}b_m \quad (n \geq 1) \iff b_n = \frac{1}{n}\sum_{m=1}^{n}\binom{-nt}{n-m}a_m \quad (n \geq 1).$$

Theorem 3.4. *Let $f(x) = c_0 + c_1x + c_2x^2 + \cdots$ with $c_0 \neq 0$. For $k, n \in \mathbb{N}$ with $k < n$ we have*

$$\sum_{m=k}^{n}\frac{1}{m}[x^{n-m}]f(x)^m \cdot [x^{m-k}]f(x)^{-m} = \sum_{m=k}^{n}m[x^{m-k}]f(x)^k \cdot [x^{n-m}]f(x)^{-n} = 0.$$

Proof. For $m \in \mathbb{N}$ let $b_m = \frac{1}{m}\sum_{k=1}^{m}[x^{m-k}]f(x)^{-m} \cdot y^k$. Applying Theorem 3.2 we see that

$$\sum_{m=1}^{n}[x^{n-m}]f(x)^m \cdot b_m = \frac{y^n}{n}.$$

On the other hand,

$$\sum_{m=1}^{n}[x^{n-m}]f(x)^m\cdot b_m=\sum_{m=1}^{n}[x^{n-m}]f(x)^m\cdot\frac{1}{m}\sum_{k=1}^{m}[x^{m-k}]f(x)^{-m}\cdot y^k$$
$$=\sum_{k=1}^{n}\Big(\sum_{m=k}^{n}[x^{n-m}]f(x)^m\cdot\frac{1}{m}[x^{m-k}]f(x)^{-m}\Big)y^k.$$

Thus,

$$\sum_{k=1}^{n}\Big(\sum_{m=k}^{n}\frac{1}{m}[x^{n-m}]f(x)^m[x^{m-k}]f(x)^{-m}\Big)y^k=\frac{y^n}{n}$$

and hence

$$\sum_{m=k}^{n}\frac{1}{m}[x^{n-m}]f(x)^m\cdot[x^{m-k}]f(x)^{-m}=0\quad\text{for}\quad k<n.$$

For $m\in\mathbb{N}$ let $a_m=m\sum_{k=1}^{m}[x^{m-k}]f(x)^k\cdot y^k$. Applying Theorem 3.2 we have

$$\sum_{m=1}^{n}[x^{n-m}]f(x)^{-n}\cdot a_m=ny^n.$$

On the other hand,

$$\sum_{m=1}^{n}[x^{n-m}]f(x)^{-n}\cdot a_m=\sum_{m=1}^{n}[x^{n-m}]f(x)^{-n}\cdot m\sum_{k=1}^{m}[x^{m-k}]f(x)^k\cdot y^k$$
$$=\sum_{k=1}^{n}\Big(\sum_{m=k}^{n}[x^{n-m}]f(x)^{-n}\cdot m[x^{m-k}]f(x)^k\Big)y^k.$$

Thus,

$$\sum_{k=1}^{n}\Big(\sum_{m=k}^{n}m[x^{m-k}]f(x)^k\cdot[x^{n-m}]f(x)^{-n}\Big)y^k=ny^n$$

and hence

$$\sum_{m=k}^{n}m[x^{m-k}]f(x)^k\cdot[x^{n-m}]f(x)^{-n}=0\quad\text{for}\quad k<n.$$

This completes the proof.

Corollary 3.1. *For $k,n\in\mathbb{N}$ with $k<n$ we have*

$$\sum_{m=k}^{n}\frac{1}{m}\binom{mt}{n-m}\binom{-mt}{m-k}=\sum_{m=k}^{n}m\binom{kt}{m-k}\binom{-nt}{n-m}=0.$$

Proof. Since $(1+x)^{rt}=\sum_{s=0}^{\infty}\binom{rt}{s}x^s$, taking $f(x)=(1+x)^t$ in Theorem 3.4 we deduce the result.

For the development of combinatorial inversion formulas, see [1-13].

4. A formula for $[x^{m+n}]\alpha(x)^m$

Theorem 4.1. *Let $\beta(x)=x\sum_{n=0}^{\infty}\beta_nx^n$ with $\beta_0\neq0$. Let $\alpha(x)$ be the inverse function of $\beta(x)$. For $m,n\in\mathbb{N}$ we have*

$$[x^{m+n}]\alpha(x)^m=\frac{m}{(m+n)!}\sum_{k_1+2k_2+\cdots+nk_n=n}\frac{(m+n-1+k_1+\cdots+k_n)!}{k_1!\cdots k_n!}$$

$$\times (-1)^{k_1+k_2+\cdots+k_n}\beta_0^{-n-m-k_1-\cdots-k_n}\beta_1^{k_1}\beta_2^{k_2}\cdots\beta_n^{k_n}.$$

Proof. By the multinomial theorem we have

$$\Big(\sum_{k=1}^{n}\frac{\beta_k}{\beta_0}x^k\Big)^s = \sum_{k_1+\cdots+k_n=s}\frac{s!}{k_1!\cdots k_n!}\prod_{i=1}^{n}\Big(\frac{\beta_i}{\beta_0}x^i\Big)^{k_i}.$$

Thus

$$[x^n]\Big(\sum_{k=1}^{\infty}\frac{\beta_k}{\beta_0}x^k\Big)^s = [x^n]\Big(\sum_{k=1}^{n}\frac{\beta_k}{\beta_0}x^k\Big)^s = \sum_{\substack{k_1+\cdots+k_n=s\\ k_1+2k_2+\cdots+nk_n=n}}\frac{s!}{k_1!\cdots k_n!}\prod_{i=1}^{n}\Big(\frac{\beta_i}{\beta_0}\Big)^{k_i}.$$

As

$$\begin{aligned}
&\beta_0^{m+n}\Big(\frac{x}{\beta(x)}\Big)^{m+n}-1\\
&=\beta_0^{m+n}\Big(\beta_0+\sum_{k=1}^{\infty}\beta_kx^k\Big)^{-n-m}-1=\Big(1+\sum_{k=1}^{\infty}\frac{\beta_k}{\beta_0}x^k\Big)^{-n-m}-1\\
&=\sum_{s=1}^{\infty}\frac{(-n-m)(-n-m-1)\cdots(-n-m-s+1)}{s!}\Big(\sum_{k=1}^{\infty}\frac{\beta_k}{\beta_0}x^k\Big)^s,
\end{aligned}$$

from the above we see that

$$\begin{aligned}
&[x^n]\beta_0^{m+n}\Big(\frac{x}{\beta(x)}\Big)^{m+n}\\
&=\sum_{s=1}^{\infty}\frac{(-n-m)(-n-m-1)\cdots(-n-m-s+1)}{s!}\\
&\qquad\times\sum_{\substack{k_1+\cdots+k_n=s\\ k_1+2k_2+\cdots+nk_n=n}}\frac{s!}{k_1!\cdots k_n!}\prod_{i=1}^{n}\Big(\frac{\beta_i}{\beta_0}\Big)^{k_i}\\
&=\sum_{k_1+2k_2+\cdots+nk_n=n}\frac{(m+n)(m+n+1)\cdots(m+n+k_1+\cdots+k_n-1)}{k_1!\cdots k_n!}\\
&\qquad\times\Big(-\frac{1}{\beta_0}\Big)^{k_1+\cdots+k_n}\beta_1^{k_1}\cdots\beta_n^{k_n}.
\end{aligned}$$

Thus applying Lemma 3.2 we have

$$\begin{aligned}
[x^{m+n}]\alpha(x)^m &= \frac{m}{m+n}[x^n]\Big(\frac{x}{\beta(x)}\Big)^{m+n}\\
&=\frac{m}{m+n}\beta_0^{-m-n}\sum_{k_1+2k_2+\cdots+nk_n=n}\frac{(k_1+\cdots+k_n+m+n-1)!}{k_1!\cdots k_n!(m+n-1)!}\cdot\\
&\qquad\times(-1)^{k_1+\cdots+k_n}\beta_0^{-(k_1+\cdots+k_n)}\beta_1^{k_1}\cdots\beta_n^{k_n}.
\end{aligned}$$

This yields the result.

Corollary 4.1. *For $m,n\in\mathbb{N}$ we have*

$$\sum_{k_1+2k_2+\cdots+nk_n=n}\frac{(k_1+\cdots+k_n+m+n-1)!}{(m+n-1)!k_1!\cdots k_n!}(-1)^{k_1+\cdots+k_n}=(-1)^n\binom{m+n}{m}.$$

Proof. Let $\beta(x)=x\sum_{r=0}^{\infty}x^r=\frac{x}{1-x}$. Then the inverse function of $\beta(x)$ is given by $\alpha(x)=\frac{x}{1+x}$. Using the binomial theorem we see that $[x^{m+n}]\alpha(x)^m=[x^n](1+x)^{-m}=\binom{-m}{n}=(-1)^n\binom{m+n-1}{n}$. Now applying Theorem 4.1 we deduce the result.

Corollary 4.2. *For $n \in \mathbb{N}$ we have*

$$\sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+n)!}{k_1!\cdots k_n!}(-1)^{k_1+\cdots+k_n}2^{k_1}3^{k_2}\cdots(n+1)^{k_n}$$
$$= (-1)^n \cdot (n+1)! \cdot \frac{1}{n+2}\binom{2n+2}{n+1}.$$

Proof. Let

$$\beta(x) = \frac{x}{(1+x)^2} \quad \text{and} \quad \alpha(x) = \frac{1-\sqrt{1-4x}}{2x} - 1 \ \big(0 < x < \frac{1}{4}\big).$$

It is easily seen that $\alpha(x) = \beta^{-1}(x)$. From [5, p.53] and the binomial theorem we know that

$$\alpha(x) = x\sum_{n=0}^{\infty} \frac{1}{n+2}\binom{2n+2}{n+1}x^n \quad \text{and} \quad \beta(x) = x\sum_{n=0}^{\infty}(-1)^n(n+1)x^n.$$

Now applying Theorem 4.1 (with $m = 1$) we deduce the result.

Corollary 4.3. *For $n \in \mathbb{N}$ we have*

$$\sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+2n)!}{k_1!\cdots k_n!} \cdot \frac{(-1)^{k_1+k_2+\cdots+k_n+n}}{3!^{k_1}5!^{k_2}\cdots(2n+1)!^{k_n}} = (2n-1)!!^2.$$

Proof. It is well known that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

and

$$\arcsin x = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}(2n+1)}x^{2n+1} \ (|x| < 1).$$

Set $\beta(x) = \sin x = x\sum_{n=0}^{\infty}\beta_n x^n$. Then $\beta^{-1}(x) = \arcsin x$ and

$$\beta_i = \begin{cases} 0 & \text{if } 2 \nmid i, \\ \dfrac{(-1)^{i/2}}{(i+1)!} & \text{if } 2 \mid i. \end{cases}$$

Thus, taking $m = 1$ and $\alpha(x) = \arcsin x$ in Theorem 4.1 and substituting n by $2n$ we obtain

$$\begin{aligned} &(2n+1)! \cdot [x^{2n+1}] \arcsin x \\ &= \sum_{k_1+2k_2+\cdots+2nk_{2n}=2n} \frac{(2n+k_1+k_2+\cdots+k_{2n})!}{k_1!k_2!\cdots k_{2n}!}(-1)^{k_1+k_2+\cdots+k_{2n}}\beta_1^{k_1}\beta_2^{k_2}\cdots\beta_{2n}^{k_{2n}} \\ &= \sum_{k_2+2k_4+\cdots+nk_{2n}=n} \frac{(2n+k_2+k_4+\cdots+k_{2n})!}{k_2!k_4!\cdots k_{2n}!}(-1)^{k_2+k_4+\cdots+k_{2n}}\prod_{i=1}^{n}\Big(\frac{(-1)^i}{(2i+1)!}\Big)^{k_{2i}}. \end{aligned}$$

Replacing k_{2i} with k_i in the above formula and observing that

$$(2n+1)! \cdot [x^{2n+1}] \arcsin x = (2n+1)! \cdot \frac{\binom{2n}{n}}{2^{2n}(2n+1)} = (2n-1)!!^2$$

we deduce the result.

Theorem 4.2. *For $m, n \in \mathbb{N}$ we have*

$$S(m+n,m) = \frac{1}{(m-1)!} \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+\cdots+k_n+n} \frac{(k_1+\cdots+k_n+m+n-1)!}{2^{k_1}k_1! \cdot 3^{k_2}k_2! \cdots (n+1)^{k_n}k_n!}$$

and

$$s(m+n,m) = \frac{1}{(m-1)!} \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+\cdots+k_n+n} \frac{(k_1+\cdots+k_n+m+n-1)!}{2!^{k_1}k_1! \cdot 3!^{k_2}k_2! \cdots (n+1)!^{k_n}k_n!}.$$

Proof. Clearly $e^x - 1$ and $\log(1+x)$ are a pair of inverse functions. As

$$\frac{(e^x-1)^m}{m!} = \sum_{n=0}^{\infty} S(m+n,m)\frac{x^{m+n}}{(m+n)!} \quad \text{and} \quad \log(1+x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} x^{i+1},$$

putting $\alpha(x) = e^x - 1$, $\beta(x) = \log(1+x)$ and $\beta_i = \frac{(-1)^i}{i+1}$ in Theorem 4.1 we see that

$$\begin{aligned}\frac{m!S(m+n,m)}{(m+n)!} &= [x^{m+n}](e^x-1)^m \\ &= \frac{m}{(m+n)!} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+m+n-1)!}{k_1!\cdots k_n!} \\ &\quad \times (-1)^{k_1+k_2+\cdots+k_n} \cdot (-1)^{k_1+2k_2+\cdots+nk_n} \frac{1}{2^{k_1} \cdot 3^{k_2} \cdots (n+1)^{k_n}}.\end{aligned}$$

Since

$$\frac{(\log(1+x))^m}{m!} = \sum_{n=0}^{\infty}(-1)^n s(m+n,m)\frac{x^{m+n}}{(m+n)!} \quad \text{and} \quad e^x - 1 = \sum_{i=0}^{\infty} \frac{x^{i+1}}{(i+1)!},$$

putting $\alpha(x) = \log(1+x)$, $\beta(x) = e^x - 1$ and $\beta_i = \frac{1}{(i+1)!}$ in Theorem 4.1 we see that

$$\begin{aligned}&(-1)^n \frac{m!s(m+n,m)}{(m+n)!} \\ &= [x^{m+n}](\log(1+x))^m \\ &= \frac{m}{(m+n)!} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+m+n-1)!}{k_1!\cdots k_n!} \cdot \frac{(-1)^{k_1+\cdots+k_n}}{2!^{k_1} \cdot 3!^{k_2} \cdots (n+1)!^{k_n}}.\end{aligned}$$

By the above, the theorem is proved.

We remark that Theorem 4.2 provides a straightforward method to calculate $s(m+n,m)$ and $S(m+n,m)$ for small n. For example, we have

$$S(m+3,m) = \binom{m+1}{2}\binom{m+3}{4} \quad \text{and} \quad s(m+3,m) = \binom{m+3}{2}\binom{m+3}{4}. \tag{4.1}$$

Corollary 4.4. *For $m, n \in \mathbb{N}$ we have*

$$\begin{aligned}&\sum_{r=0}^{m} \binom{m}{r}(-1)^{m-r} r^{m+n} \\ &= m \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+\cdots+k_n+n} \frac{(k_1+\cdots+k_n+m+n-1)!}{2^{k_1}k_1! \cdot 3^{k_2}k_2! \cdots (n+1)^{k_n}k_n!}.\end{aligned}$$

Proof. It is well known that ([5, p.204])

$$\sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r} r^{m+n} = m! S(m+n, m).$$

Combining this with Theorem 4.2 we obtain the result.

Let $\alpha(x) = -x + \alpha_1 x^2 + \alpha_2 x^3 + \cdots$ and $\beta(x) = -x + \beta_1 x^2 + \beta_2 x^3 + \cdots$ be a pair of inverse functions. Taking $m = 1$ in Theorem 4.1 we deduce:

Theorem 4.3. *We have the following inversion formula:*

$$\begin{aligned}\alpha_n &= \frac{(-1)^{n+1}}{(n+1)!} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+n)!}{k_1!\cdots k_n!} \beta_1^{k_1}\cdots\beta_n^{k_n} (n \geq 1) \\ \Longleftrightarrow\ \beta_n &= \frac{(-1)^{n+1}}{(n+1)!} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+n)!}{k_1!\cdots k_n!} \alpha_1^{k_1}\cdots\alpha_n^{k_n} (n \geq 1).\end{aligned}$$

Definition 4.1. *If $\alpha(x) = \alpha^{-1}(x)$, we say that $\alpha(x)$ is a self-inverse function.*

For example, $\alpha(x) = \frac{rx+s}{tx-r}$ $((r^2+t^2)(r^2+st) \neq 0)$ and $\alpha(x) = (1-x^k)^{\frac{1}{k}}$ are self-inverse functions.

Theorem 4.4. *Let $\alpha(x) = -x+\alpha_1 x^2+\alpha_2 x^3+\cdots$ be a self-inverse function. Then $\alpha_2, \alpha_4, \ldots$ depend only on $\alpha_1, \alpha_3, \ldots$. Moreover, for $n \in \mathbb{N}$,*

$$\begin{aligned}&\sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n} \frac{(k_1+\cdots+k_{n-1}+n)!}{k_1!\cdots k_{n-1}!} \alpha_1^{k_1}\cdots\alpha_{n-1}^{k_{n-1}} \\ &= \begin{cases} 0 & \text{if } 2 \nmid n, \\ -2\cdot(n+1)!\alpha_n & \text{if } 2 \mid n. \end{cases}\end{aligned} \tag{4.2}$$

Proof. By Theorem 4.3 we have

$$\begin{aligned}\alpha_n &= \frac{(-1)^{n+1}}{(n+1)!} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+n)!}{k_1!\cdots k_n!} \alpha_1^{k_1}\cdots\alpha_n^{k_n} \\ &= \frac{(-1)^{n+1}}{(n+1)!} \sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n} \frac{(k_1+\cdots+k_{n-1}+n)!}{k_1!\cdots k_{n-1}!} \alpha_1^{k_1}\cdots\alpha_{n-1}^{k_{n-1}} + (-1)^{n+1}\alpha_n.\end{aligned}$$

Thus (4.2) is true. Using (4.2) and induction we deduce that $\alpha_2, \alpha_4, \ldots$ depend only on $\alpha_1, \alpha_3, \ldots$. This completes the proof.

If $\alpha(x) = -x + \alpha_1 x^2 + \alpha_2 x^3 + \cdots$ is a self-inverse function, from (4.2) we deduce

$$\begin{aligned}&\alpha_2 = -\alpha_1^2,\ \alpha_4 = 2\alpha_1^4 - 3\alpha_1\alpha_3, \\ &\alpha_6 = -13\alpha_1^6 - 4\alpha_1\alpha_5 - 2\alpha_3^2 + 18\alpha_1^3\alpha_3, \\ &\alpha_8 = 145\alpha_1^8 - 221\alpha_1^5\alpha_3 + 50\alpha_1^2\alpha_3^2 + 35\alpha_1^3\alpha_5 - 5\alpha_3\alpha_5 - 5\alpha_1\alpha_7.\end{aligned} \tag{4.3}$$

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