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Ramsey numbers for trees

Zhi-Hong Sun
School of Mathematical Sciences,
Huaiyin Normal University,
Huaian, Jiangsu 223001, P.R. China
E-mail: zhihongsun@yahoo.com
Homepage: http://www.hytc.edu.cn/xsjl/szh

Abstract

For $n \ge 5$ let T_n' denote the unique tree on n vertices with $\Delta(T_n') = n-2$, and let $T_n^* = (V, E)$ be the tree on n vertices with $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In this paper we evaluate the Ramsey numbers $r(G_m, T_n')$ and $r(G_m, T_n^*)$, where G_m is a connected graph of order m. As examples, for $n \ge 8$ we have $r(T_n', T_n^*) = r(T_n^*, T_n^*) = 2n-5$, for $n > m \ge 7$ we have $r(K_{1,m-1}, T_n^*) = m+n-3$ or $m+n-4$ according as $m-1 \mid (n-3)$ or $m-1 \nmid (n-3)$, for $m \ge 7$ and $n \ge (m-3)^2+2$ we have $r(T_m^*, T_n^*) = m+n-3$ or $m+n-4$ according as $m-1 \mid (n-3)$ or $m-1 \nmid (n-3)$.

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1. Introduction

In this paper, all graphs are simple graphs. For a graph $G = (V(G), E(G))$ let $e(G) = |E(G)|$ be the number of edges in G and let $\Delta(G)$ be the maximal degree of G. For a forbidden graph L, let $ex(p; L)$ denote the maximal number of edges in a graph of order p not containing L as a subgraph. The corresponding Turán's problem is to evaluate $ex(p; L)$.

Let $\mathbb{N}$ be the set of positive integers, and let $p, n \in \mathbb{N}$ with $p \ge n \ge 3$. For a given tree T_n on n vertices, it is difficult to determine the value of $ex(p; T_n)$. The famous Erdös-Sós conjecture asserts that $ex(p; T_n) \le \frac{(n-2)p}{2}$ for every tree T_n on n vertices. For the progress on the Erdös-Sós conjecture, see [4,8,9,11]. Write $p = k(n-1)+r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n-2\}$. Let P_n be the path on n vertices. In [5] Faudree and Schelp showed that

$$ex(p; P_n) = k\binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}. \qquad (1.1)$$

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In the special case $r = 0$, (1.1) is due to Erdös and Gallai [3]. Let $K_{1,n-1}$ denote the unique tree on n vertices with $\Delta(K_{1,n-1}) = n-1$, and for $n \geq 4$ let T'_n denote the unique tree on n vertices with $\Delta(T'_n) = n-2$. In [10] the author and Lin-Lin Wang obtained exact values of $ex(p; K_{1,n-1})$ and $ex(p; T'_n)$, see Lemmas 2.4 and 2.5.

For $n \geq 5$ let $T^*_n = (V, E)$ be the tree on n vertices with $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In [10], we also determine the value of $ex(p; T^*_n)$, see Lemmas 2.6-2.8.

As usual $\overline{G}$ denotes the complement of a graph G. Let G_1 and G_2 be two graphs. The Ramsey number $r(G_1, G_2)$ is the smallest positive integer n such that, for every graph G with n vertices, either G contains a copy of G_1 or else $\overline{G}$ contains a copy of G_2.

Let $n \in \mathbb{N}$ with $n \geq 6$. If the Erdös-Sós conjecture is true, it is known that $r(T_n, T_n) \leq 2n-2$ (see [8]). Let $m, n \in \mathbb{N}$. In 1973 Burr and Roberts[2] showed that for $m, n \geq 3$,

$$r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m+n-3 & \text{if } 2 \nmid mn, \\ m+n-2 & \text{if } 2 \mid mn. \end{cases}$$

In 1995, Guo and Volkmann[6] proved that for $n \geq m \geq 5$,

$$r(T'_m, T'_n) = \begin{cases} m+n-3 & \text{if } m-1 \mid (n-3), \\ m+n-5 & \text{if } m = n \equiv 0 \pmod 2, \\ m+n-4 & \text{otherwise} \end{cases}$$

and, for $n > m \geq 4$,

$$r(K_{1,m-1}, T'_n) = \begin{cases} m+n-3 & \text{if } 2 \mid m(n-1), \\ m+n-4 & \text{if } 2 \nmid m(n-1). \end{cases}$$

Let $m, n \in \mathbb{N}$ with $n \geq m \geq 6$. In this paper we evaluate the Ramsey number $r(T_m, T^*_n)$ for $T_m \in \{P_m, K_{1,m-1}, T'_m, T^*_m\}$. As examples, for $n \geq 8$,

$$r(P_n, T^*_n) = r(T^*_n, T^*_n) = 2n-5;$$

for $n > m \geq 7$,

$$r(K_{1,m-1}, T^*_n) = \begin{cases} m+n-3 & \text{if } m-1 \mid (n-3), \\ m+n-4 & \text{if } m-1 \nmid (n-3); \end{cases}$$

and, for $m \geq 7$ and $n \geq (m-3)^2 + 2$,

$$r(P_m, T^*_n) = r(T'_m, T^*_n) = r(T^*_m, T^*_n) = \begin{cases} m+n-3 & \text{if } m-1 \mid (n-3), \\ m+n-4 & \text{if } m-1 \nmid (n-3). \end{cases}$$

In addition to the above notation, throughout the paper we also use the following notation: $\lfloor x \rfloor$ is the greatest integer not exceeding x, K_n is the complete graph on n vertices, $K_{m,n}$ is the complete bipartite graph with m and n vertices in the bipartition, $d_G(v)$ is the degree of the vertex v in given graph G, and $d(u, v)$ is the distance between the two vertices u and v in a graph.

2. Basic lemmas

Lemma 2.1. *Let G_1 and G_2 be two graphs. Suppose $p \in \mathbb{N}, p \geq max\{|V(G_1)|, |V(G_2)|\}$ and $ex(p; G_1) + ex(p; G_2) < \binom{p}{2}$. Then $r(G_1, G_2) \leq p$.*

Proof. Let G be a graph of order p. If $e(G) \leq ex(p; G_1)$ and $e(\overline{G}) \leq ex(p; G_2)$, then

$$ex(p; G_1) + ex(p; G_2) \geq e(G) + e(\overline{G}) = \binom{p}{2}.$$

This contradicts the assumption. Hence, either $e(G) > ex(p; G_1)$ or $e(\overline{G}) > ex(p; G_2)$. Therefore, G contains a copy of G_1 or $\overline{G}$ contains a copy of G_2. This shows that $r(G_1, G_2) \leq |V(G)| = p$. So the lemma is proved.

Lemma 2.2. *Let $k, p \in \mathbb{N}$ with $p \geq k + 1$. Then there exists a $k-$regular graph of order p if and only if $2 \mid kp$.*

This is a known result; see, for example, [10, Corollary 2.1].

Lemma 2.3. *Let G_1 and G_2 be two graphs with $\Delta(G_1) = d_1 \geq 2$ and $\Delta(G_2) = d_2 \geq 2$. Then*

(i) $r(G_1, G_2) \geq d_1 + d_2 - (1 - (-1)^{(d_1-1)(d_2-1)})/2$.

(ii) *Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \leq m$. Then $r(G_1, G_2) \geq 2d_2 - 1 \geq d_1 + d_2$.*

(iii) *Suppose that G_1 is a connected graph of order m and $d_2 > m$. If one of the conditions*

(1) $2 \mid (d_1 + d_2 - m)$,

(2) $d_1 \neq m - 1$,

(3) *G_2 has two vertices u and v such that $d(v) = \Delta(G_2)$ and $d(u, v) = 3$*

holds, then $r(G_1, G_2) \geq d_1 + d_2$.

Proof. We first consider (i). If $2 \mid (d_1 - 1)(d_2 - 1)$, then $2 \mid (d_1 - 1)(d_1 + d_2 - 1)$. Since $d_1 - 1 \geq 1$, by Lemma 2.2 we may construct a $d_1 - 1$-regular graph G of order $d_1 + d_2 - 1$. Since $\Delta(G) = d_1 - 1$ and $\Delta(\overline{G}) = d_2 - 1$, G does not contain G_1 as a subgraph and $\overline{G}$ does not contain G_2 as a subgraph. Hence $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2$. Now we assume $2 \nmid (d_1 - 1)(d_2 - 1)$. Then $2 \mid d_1$, $2 \mid d_2$ and so $2 \mid (d_1 + d_2 - 2)$. By Lemma 2.2, we may construct a $d_1 - 1$-regular graph G of order $d_1 + d_2 - 2$. Since $\Delta(G) = d_1 - 1$ and $\Delta(\overline{G}) = d_2 - 2$, G does not contain G_1 as a subgraph and $\overline{G}$ does not contain G_2 as a subgraph. Hence $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2 - 1$. This proves (i).

Next we consider (ii). Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \leq m$. Since $K_{d_2-1} \cup K_{d_2-1}$ does not contain any copies of G_1, and its complement K_{d_2-1,d_2-1} does not contain any copies of G_2, we see that $r(G_1, G_2) \geq 1 + 2(d_2 - 1) = 2d_2 - 1 \geq d_1 + d_2$. This proves (ii).

Finally we consider (iii). Suppose that G_1 is a connected graph of order m and $d_2 > m$. By Lemma 2.2, we may construct a graph

$$G = \begin{cases} K_{m-1} \cup H_1 & \text{if } 2 \mid (d_1 + d_2 - m), \\ K_{m-2} \cup H_2 & \text{if } 2 \nmid (d_1 + d_2 - m), \end{cases}$$

where H_1 is a $d_1 - 1$-regular graph of order $d_1 + d_2 - m$ and H_2 is a $d_1 - 1$-regular graph of order $d_1 + d_2 - m + 1$. It is easily seen that G does not contain any copies of G_1 and

$$\Delta(\overline{G}) = \begin{cases} d_2 - 1 & \text{if } 2 \mid (d_1 + d_2 - m) \text{ or } d_1 \neq m - 1, \\ d_2 & \text{if } 2 \nmid (d_1 + d_2 - m) \text{ and } d_1 = m - 1. \end{cases}$$

If $2 \mid (d_1 + d_2 - m)$ or $d_1 \neq m - 1$, then $\overline{G}$ does not contain any copies of G_2 and so $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2$. Now assume $2 \nmid (d_1 + d_2 - m)$ and $d_1 = m - 1$. For $v_0 \in V(H_2)$ we have $d_{\overline{G}}(v_0) = d_2 - 1$. Suppose that $v_1, \ldots, v_{m-2} \in V(G)$ and $v_1, \ldots, v_{m-2}$ induce a copy of K_{m-2}. Then $\{v_1, \ldots, v_{m-2}\}$ is an independent set in $\overline{G}$ and $d_{\overline{G}}(v_i) = d_2$ for $i = 1, 2, \ldots, m - 2$. If G_2 has two vertices u and v such that $d(v) = \Delta(G_2)$ and $d(u, v) = 3$, we see that $\overline{G}$ does not contain any copies of G_2 and so $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2$. This proves (iii) and the lemma is proved.

Lemma 2.4 ([10, Theorem 2.1]). *Let $p, n \in \mathbb{N}$ with $p \geq n - 1 \geq 1$. Then $ex(p; K_{1,n-1}) = \lfloor \frac{(n-2)p}{2} \rfloor$.*

Lemma 2.5 ([10, Theorem 3.1]). *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 5$. Let $r \in \{0, 1, \ldots, n - 2\}$ be given by $p \equiv r \pmod{n-1}$. Then*

$$ex(p; T'_n) = \begin{cases} \lfloor \dfrac{(n-2)(p-1) - r - 1}{2} \rfloor & \text{if } n \geq 7 \text{ and } 2 \leq r \leq n - 4, \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.6 ([10, Theorems 4.1-4.3]). *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 6$, and let $p = k(n-1) + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, n - 5, n - 4, n - 3, n - 2\}$. Then*

$$ex(p; T^*_n) = \begin{cases} \dfrac{(n-2)(p-2)}{2} + 1 & \text{if } n > 6 \text{ and } r = n - 5, \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.7 ([10, Theorem 4.4]). *Let $p, n \in \mathbb{N}$, $p \geq n \geq 11$, $r \in \{2, 3, \ldots, n-6\}$ and $p \equiv r \pmod{n-1}$. Let $t \in \{0, 1, \ldots, r+1\}$ be given by $n - 3 \equiv t \pmod{r+2}$. Then*

$$ex(p; T^*_n) = \begin{cases} \lfloor \dfrac{(n-2)(p-1) - 2r - t - 3}{2} \rfloor & \text{if } r \geq 4 \text{ and } 2 \leq t \leq r - 1, \\ \dfrac{(n-2)(p-1) - t(r+2-t) - r - 1}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.8 ([10, Theorem 4.5]). *Let $p, n \in \mathbb{N}$ with $6 \leq n \leq 10$ and $p \geq n$, and let $r \in \{0, 1, \ldots, n - 2\}$ be given by $p \equiv r \pmod{n-1}$.*

(i) *If $n = 6, 7$, then $ex(p; T^*_n) = \frac{(n-2)p - r(n-1-r)}{2}$.*

(ii) *If $n = 8, 9$, then*

$$ex(p; T^*_n) = \begin{cases} \dfrac{(n-2)p - r(n-1-r)}{2} & \text{if } r \neq n - 5, \\ \dfrac{(n-2)(p-2)}{2} + 1 & \text{if } r = n - 5. \end{cases}$$

(iii) *If $n = 10$, then*

$$ex(p; T^*_n) = \begin{cases} 4p - \dfrac{r(9-r)}{2} & \text{if } r \neq 4, 5, \\ 4p - 7 & \text{if } r = 5, \\ 4p - 9 & \text{if } r = 4. \end{cases}$$

Lemma 2.9. *Let $p, m \in \mathbb{N}$ with $p \geq m \geq 5$, and $T_m \in \{P_m, K_{1,m-1}, T'_m, T^*_m\}$. Then $ex(p; T_m) \leq \frac{(m-2)p}{2}$. Moreover, if $m-1 \nmid p$ and $T_m \in \{P_m, T'_m, T^*_m\}$, then $ex(p; T_m) \leq \frac{(m-2)(p-1)}{2}$.*

Proof. This is immediate from (1.1) and Lemmas 2.4-2.8.

Lemma 2.10. *Let $m, n \in \mathbb{N}$ with $m, n \geq 5$. Let G_m be a connected graph on m vertices. If $m + n - 5 = (m-1)x + (m-2)y$ for some nonnegative integers x and y, then $r(G_m, T_n) \geq m + n - 4$ for $T_n \in \{K_{1,n-1}, T'_n, T^*_n\}$.*

Proof. Let $G = xK_{m-1} \cup yK_{m-2}$. Then $|V(G)| = m + n - 5$, $\Delta(G) \leq m - 1$ and $\Delta(\overline{G}) \leq n - 3$. Clearly, G does not contain G_m as a subgraph, and $\overline{G}$ does not contain T_n as a subgraph. So the result is true.

Lemma 2.11 ([7, Theorem 8.3, pp.11-12]). *Let $a, b, n \in \mathbb{N}$. If a is coprime to b and $n \geq (a-1)(b-1)$, then there are two nonnegative integers x and y such that $n = ax + by$.*

Conjecture 2.12. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 5$, $p = k(n-1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n-2\}$. Let $T_n \neq K_{1,n-1}, T'_n$ be a tree on n vertices. Then $ex(p; T_n) \leq ex(p; T^*_n)$. Hence:*

(i) *if $r \in \{0, 1, n-4, n-3, n-2\}$, then*

$$ex(p; T_n) = \frac{(n-2)p - r(n-1-r)}{2}.$$

(ii) *if $2 \leq r \leq n-5$, then*

$$ex(p; T_n) \leq \frac{(n-2)(p-1) - r - 1}{2}.$$

We note that

$$ex(p; T_n) \geq e(kK_{n-1} \cup K_r) = \frac{(n-2)p - r(n-1-r)}{2} = ex(p; P_n).$$

Definition 2.13. *For $n \geq 5$ let T_n be a tree on n vertices. View T_n as a bipartite graph with s_1 and s_2 vertices in the bipartition. Define $\alpha_2(T_n) = max\ \{s_1, s_2\}$.*

Conjecture 2.14. *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 5$. Let $T^{(1)}_n$ and $T^{(2)}_n$ be two trees on n vertices. If $\alpha_2(T^{(1)}_n) < \alpha_2(T^{(2)}_n)$, then $ex(p; T^{(1)}_n) \leq ex(p; T^{(2)}_n)$.*

3. The Ramsey number $r(G_n, T^*_n)$

Lemma 3.1. *Let $n \in \mathbb{N}$, $n \geq 6$, and let G_n be a connected graph on n vertices such that $ex(2n-5; G_n) < n^2 - 5n + 4$. Then $r(G_n, T^*_n) = 2n - 5$.*

Proof. As $2K_{n-3}$ does not contain any copies of G_n and $\overline{2K_{n-3}} = K_{n-3,n-3}$ does not contain any copies of T^*_n, we see that $r(G_n, T^*_n) > 2(n-3)$. By Lemma 2.6 we have

$$ex(2n-5; T^*_n) = \frac{(n-2)(2n-5) - 3(n-4)}{2} = n^2 - 6n + 11.$$

Thus,

$$\begin{aligned} ex(2n-5; G_n) + ex(2n-5; T^*_n) &< n^2 - 5n + 4 + n^2 - 6n + 11 \\ &= 2n^2 - 11n + 15 = \binom{2n-5}{2}. \end{aligned}$$

Appealing to Lemma 2.1 we obtain $r(G_n, T_n^*) \le 2n-5$. So $r(G_n, T_n^*) = 2n-5$ as asserted.

Theorem 3.2. *Let $n \in \mathbb{N}$ with $n \ge 8$. Then*

$$r(P_n, T_n^*) = r(T_n', T_n^*) = r(T_n^*, T_n^*) = 2n-5.$$

Proof. By Lemma 2.6,

$$ex(2n-5; T_n^*) = \frac{(n-2)(2n-5)-3(n-4)}{2} = n^2-6n+11 < n^2-5n+4.$$

By Lemma 2.5,

$$\begin{aligned} ex(2n-5; T_n') &= \Big\lfloor \frac{(n-2)(2n-6)-(n-4)-1}{2} \Big\rfloor = \Big\lfloor n^2 - \frac{11}{2}n + \frac{15}{2} \Big\rfloor \\ &\le n^2 - \frac{11}{2}n + \frac{15}{2} < n^2-5n+4. \end{aligned}$$

By (1.1),

$$ex(2n-5; P_n) = \binom{n-1}{2} + \binom{n-4}{2} = n^2-6n+11 < n^2-5n+4.$$

Thus applying Lemma 3.1 we deduce the result.

Conjecture 3.3. *Let $n \in \mathbb{N}$, $n \ge 8$, and let $T_n \ne K_{1,n-1}$ be a tree on n vertices. Then $r(T_n, T_n^*) = 2n-5$.*

Remark 3.4 Let $n \in \mathbb{N}$ with $n \ge 4$. From [6, Theorem 3.1(ii)] we know that $r(K_{1,n-1}, T_n^*) = 2n-3$.

4. The Ramsey number $r(G_m, T_n^*)$ for $m < n$

Theorem 4.1. *Let $m, n \in \mathbb{N}$, $n > m \ge 5$ and $m-1 \mid n-3$. Let G_m be a connected graph of order m such that $ex(m+n-3; G_m) \le \frac{(m-2)(m+n-3)}{2}$ or $G_m \in \{P_m, K_{1,m-1}, T_m', T_m^*\}$. Then $r(G_m, T_n^*) = m+n-3$.*

Proof. By Lemma 2.9 we may assume that $ex(m+n-3; G_m) \le \frac{(m-2)(m+n-3)}{2}$. Suppose that $n-3 = k(m-1)$. Clearly $(k+1)K_{m-1}$ does not contain G_m as a subgraph and $\overline{(k+1)K_{m-1}}$ does not contain T_n^* as a subgraph. Thus

$$r(G_m, T_n^*) > (k+1)(m-1) = m+n-4.$$

Since $1 \le m-4 \le n-6$, using Lemma 2.9 we see that

$$ex(m+n-3; T_n^*) \le \frac{(n-2)(m+n-4)}{2}.$$

Thus,

$$\begin{aligned} &ex(m+n-3; G_m) + ex(m+n-3; T_n^*) \\ &\le \frac{(m-2)(m+n-3)}{2} + \frac{(n-2)(m+n-4)}{2} \\ &< \frac{(m-2+n-2)(m+n-3)}{2} = \binom{m+n-3}{2}. \end{aligned}$$

Hence, by Lemma 2.1, $r(G_m, T_n^*) \leq m+n-3$, and the result follows.

Lemma 4.2. *Let $m, n \in \mathbb{N}, n > m \geq 7$ and $m-1 \nmid n-3$. Let G_m be a connected graph of order m such that $ex(m+n-4; G_m) \leq \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{P_m, K_{1,m-1}, T'_m, T^*_m\}$. Then $r(G_m, T_n^*) \leq m+n-4$.*

Proof. By Lemma 2.9, we may assume that $ex(m+n-4; G_m) \leq \frac{(m-2)(m+n-4)}{2}$. As $m+n-4 = n-1+m-3$ and $m-1 \nmid (n-3)$, we see that $2 \leq m-3 \leq n-4$ and $m-3 \neq n-5$. Thus, applying Lemmas 2.6- 2.8,

$$ex(m+n-4; T_n^*) < \frac{(n-3)(m+n-4)}{2}.$$

Hence,

$$\begin{aligned} &ex(m+n-4; G_m) + ex(m+n-4; T_n^*) \\ &< \frac{(m-2)(m+n-4)}{2} + \frac{(n-3)(m+n-4)}{2} = \binom{m+n-4}{2}. \end{aligned}$$

Applying Lemma 2.1, we obtain the result.

Theorem 4.3. *Let $m, n \in \mathbb{N}, n > m \geq 7$ and $m-1 \nmid (n-3)$. Let G_m be a connected graph of order m such that $ex(m+n-4; G_m) \leq \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{P_m, T'_m, T^*_m\}$. If $m+n-5 = (m-1)x + (m-2)y$ for some $x, y \in \{0, 1, 2, \ldots\}$, then $r(G_m, T_n^*) = m+n-4$.*

Proof. By Lemma 4.2, $r(G_m, T_n^*) \leq m+n-4$, and by Lemma 2.10, $r(G_m, T_n^*) \geq m+n-4$. Thus the result follows.

Theorem 4.4. *Suppose $m, n \in \mathbb{N}$, $n > m \geq 7$, $n = k(m-1)+b = q(m-2)+a$, $k, q \in \mathbb{N}$, $a \in \{0, 1, \ldots, m-3\}$ and $b \in \{0, 1, \ldots, m-2\} - \{3\}$. Let G_m be a connected graph of order m such that $ex(m+n-4; G_m) \leq \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{P_m, T'_m, T^*_m\}$. If one of the conditions:*

(i) $b \in \{1, 2, 4\}$,
(ii) $b = 0$ *and* $k \geq 3$,
(iii) $n \geq (m-3)^2 + 2$,
(iv) $n \geq m^2 - 1 - b(m-2)$,
(v) $a \geq 3$ *and* $n \geq (a-4)(m-1) + 4$

holds, then $r(G_m, T_n^*) = m+n-4$.

Proof. For $b \in \{1, 2, 4\}$,

$$m+n-5 = \begin{cases} (k-2)(m-1) + 3(m-2) & \text{if } b = 1, \\ (k-1)(m-1) + 2(m-2) & \text{if } b = 2, \\ (k+1)(m-1) & \text{if } b = 4. \end{cases}$$

For $b = 0$ and $k \geq 3$ we have $m+n-5 = (k-3)(m-1)+4(m-2)$. For $n \geq (m-3)^2+2$, we have $m+n-5 \geq (m-2)(m-3)$ and so $m+n-5 = (m-1)x+(m-2)y$ for some $x, y \in \{0, 1, 2, \ldots\}$ by Lemma 2.11. For $n \geq m^2-1-b(m-2)$ we have $k \geq m+1-b$ and $m+n-5 = (k+b-m-1)(m-1) + (m+3-b)(m-2)$. For $a \geq 3$ and $n \geq (a-4)(m-1)+4$ we have $q \geq a-4$ and $m+n-5 = (a-3)(m-1)+(q+4-a)(m-2)$. Combining all the above with Theorem 4.3, we obtain the result.

Theorem 4.5. *Suppose that $m, n \in \mathbb{N}$, $n > m \geq 7$ and $m - 1 \nmid n - 3$. Then*

$$r(K_{1,m-1}, T_n^*) = m + n - 4,$$
$$r(T'_m, T_n^*) = m + n - 4 \text{ or } m + n - 5,$$
$$m + n - 6 \leq r(T_m^*, T_n^*) \leq m + n - 4.$$

Proof. From Lemma 4.2, $r(T_m, T_n^*) \leq m+n-4$ for $T_m \in \{K_{1,m-1}, T'_m, T_m^*\}$. By Lemma 2.3, $r(K_{1,m-1}, T_n^*) \geq m-1+n-3$, $r(T'_m, T_n^*) \geq m-2+n-3$ $(n > m+1)$ and $r(T_m^*, T_n^*) \geq m-3+n-3$. By Theorem 4.4, $r(T'_m, T_n^*) = m+n-4$ for $n = m+1, m+3$. Thus the theorem is proved.

Theorem 4.6. *Suppose that $m, n \in \mathbb{N}$, $n > m \geq 7$, $n = k(m-1)+b$, $k \in \mathbb{N}$, $b \in \{0, 1, \ldots, m-2\}$, $b \neq 3$ and $\frac{m-b}{2} \leq k \leq m+2-b$. Let G_m be a connected graph of order m such that $ex(m+n-4; G_m) \leq \frac{1}{2}(m-2)(m+n-4)$ or $G_m \in \{P_m, T_m^*\}$. Then $r(G_m, T_n^*) = m+n-4$ or $m+n-5$.*

Proof. By Lemma 4.2 we only need to show that $r(G_m, T_n^*) > m+n-6$. Set $G = (2k+b-m)K_{m-2} \cup (m+2-b-k)K_{m-3}$. Then $|V(G)| = (2k+b-m)(m-2) + (m+2-b-k)(m-3) = m+n-6$. We also have $\Delta(G) \leq m-2$ and $\Delta(\overline{G}) \leq m+n-6-(m-3) = n-3$. Now it is clear that G_m is not a subgraph of G and that T_n^* is not a subgraph of $\overline{G}$. So $r(G_m, T_n^*) > |V(G)|$, which completes the proof.

Remark 4.7 If $p \geq m \geq 6$ and T_m is a tree on m vertices with a vertex adjacent to at least $\lfloor \frac{m-1}{2} \rfloor$ vertices of degree 1, in [9] Sidorenko proved that $ex(p; T_m) \leq \frac{(m-2)p}{2}$. Thus, G_m can be replaced with T_m in Lemma 4.2, Theorems 4.1, 4.3, 4.4 and 4.6.

5. The Ramsey number $r(G_m, T'_n)$ for $m < n$

Theorem 5.1. *Let $m, n \in \mathbb{N}$, $n > m \geq 6$ and $m-1 \mid n-3$. Suppose that G_m is a connected graph of order m satisfying $ex(m+n-3; G_m) \leq \frac{(m-2)(m+n-3)+m+n-4}{2}$ or $G_m \in \{T_m^*, P_m\}$. Then $r(G_m, T'_n) = m+n-3$.*

Proof. By Lemma 2.9 we may assume that

$$ex(m+n-3; G_m) \leq (m-2)(m+n-3)/2 + (m+n-4)/2.$$

Suppose $n-3 = k(m-1)$ and $G = (k+1)K_{m-1}$. Then $|V(G)| = m+n-4$ and $\Delta(\overline{G}) = n-3$. Clearly, G_m is not a subgraph of G and T'_n is not a subgraph of $\overline{G}$. Thus $r(G_m, T'_n) > m+n-4$. Since $m-1 \mid (n-3)$, we have $n \geq m+2$ and so $4 \leq m-2 \leq n-4$. Hence, using Lemma 2.5, $ex(m+n-3; T'_n) = \lfloor \frac{(n-2)(m+n-4)-(m-1)}{2} \rfloor < \frac{(n-2)(m+n-3)-(m+n-4)}{2}$. Therefore

$$ex(m+n-3; G_m) + ex(m+n-3; T'_n) < \binom{m+n-3}{2}.$$

Applying Lemma 2.1, we see that $r(G_m, T'_n) \leq m+n-3$, so the result follows.

Lemma 5.2. *Let $m, n \in \mathbb{N}, n > m \geq 6$ and $m-1 \nmid n-3$. Suppose that G_m is a connected graph of order m satisfying $ex(m+n-4; G_m) < \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{T_m^*, P_m\}$. Then $r(G_m, T_n') \leq m+n-4$.*

Proof. Since $m-1 \nmid n-3$, $m-1 \nmid m+n-4$. Thus, applying Lemma 2.9, $ex(m+n-4; T_m^*) \leq (m-2)(m+n-5)/2$ and $ex(m+n-4; P_m) \leq (m-2)(m+n-5)/2$. As $n > m$, $3 \leq m-3 \leq n-4$. By Lemma 2.5, $ex(m+n-4; T_n') = \lfloor \frac{(n-2)(m+n-5)-(m-2)}{2} \rfloor \leq \frac{(n-2)(m+n-5)-(m-2)}{2}$. Thus

$$ex(m+n-4; G_m) + ex(m+n-4; T_n') < \frac{(m-2+n-2)(m+n-5)}{2} = \binom{m+n-4}{2}.$$

This, together with Lemma 2.1, yields the result.

Theorem 5.3. *Let $m, n \in \mathbb{N}$, $n > m \geq 6$ and $m-1 \nmid (n-3)$. Then $r(T_m^*, T_{m+1}') = 2m-3$ and $r(T_m^*, T_n') = m+n-4$ or $m+n-5$ for $n \geq m+3$. Suppose that G_m is a connected graph of order m satisfying $ex(m+n-4; G_m) < \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{T_m^*, P_m\}$. If $m+n-5 = (m-1)x + (m-2)y$ for some nonnegative integers x and y, then $r(G_m, T_n') = m+n-4$.*

Proof. By Lemma 2.3, $r(T_m^*, T_{m+1}') \geq 2(m-1)-1 = 2m-3$ and $r(T_m^*, T_n') \geq m-3+n-2$ for $n \geq m+3$. By Lemma 5.2, $r(G_m, T_n') \leq m+n-4$. Thus, $r(T_m^*, T_{m+1}') = 2m-3$. Applying Lemma 2.10 we deduce the remaining result.

From Theorem 5.3 and the proof of Theorem 4.4 we deduce the following result.

Theorem 5.4. *Suppose $m, n \in \mathbb{N}$, $n > m \geq 6$, $n = k(m-1)+b = q(m-2)+a$, $k, q \in \mathbb{N}$, $a \in \{0, 1, \ldots, m-3\}$ and $b \in \{0, 1, \ldots, m-2\} - \{3\}$. Let G_m be a connected graph of order m such that $ex(m+n-4; G_m) < \frac{(m-2)(m+n-4)}{2}$ or $G_m \in \{P_m, T_m^*\}$. If one of the conditions:*

(i) $b \in \{1, 2, 4\}$,

(ii) $b = 0$ *and* $k \geq 3$,

(iii) $n \geq (m-3)^2 + 2$,

(iv) $n \geq m^2 - 1 - b(m-2)$,

(v) $a \geq 3$ *and* $n \geq (a-4)(m-1) + 4$

holds, then $r(G_m, T_n') = m+n-4$.

6. The Ramsey number $r(T_m, K_{1,n-1})$ for $m < n$

The following two propositions are known.

Proposition 6.1 ([1]). *Let $m, n \in \mathbb{N}$ with $m \geq 3$ and $m-1 \mid n-2$. Let T_m be a tree on m vertices. Then $r(T_m, K_{1,n-1}) = m+n-2$.*

Proposition 6.2 ([6, Theorem 3.1]). *Let $m, n \in \mathbb{N}, m \geq 3$ and $n = k(m-1)+b$ with $k \in \mathbb{N}$ and $b \in \{0, 1, \ldots, m-2\} - \{2\}$. Let $T_m \neq K_{1,m-1}$ be a tree on m vertices. Then $r(T_m, K_{1,n-1}) \leq m+n-3$. Moreover, if $k \geq m-b$, then $r(T_m, K_{1,n-1}) = m+n-3$.*

Theorem 6.3. *Let $m, n \in \mathbb{N}$, $n \geq m \geq 3$, $m-1 \nmid (n-2)$, $n = q(m-2)+a$, $q \in \mathbb{N}$ and $a \in \{2, 3, \ldots, m-3\}$. Let $T_m \neq K_{1,m-1}$ be a tree on m vertices. If $n \geq (a-3)(m-1)+3$, then $r(T_m, K_{1,n-1}) = m+n-3$.*

Proof. Since $q(m-2) = n-a \geq (a-3)(m-2)$ we have $q \geq a-3$. Set $G = (a-2)K_{m-1} \cup (q-(a-3))K_{m-2}$. Then $|V(G)| = (a-2)(m-1)+(q-(a-3))(m-2) = m+n-4$ and $\Delta(\overline{G}) \leq n-2$. Clearly, T_m is not a subgraph of G and $K_{1,n-1}$ is not a subgraph of $\overline{G}$. Thus $r(T_m, K_{1,n-1}) > |V(G)| = m+n-4$. By Proposition 6.2, $r(T_m, K_{1,n-1}) \leq m+n-3$. So $r(T_m, K_{1,n-1}) = m+n-3$. This proves the theorem.

Theorem 6.4. *Let $m, n \in \mathbb{N}$ with $n > m \geq 5$ and $m-1 \nmid (n-2)$. Then $r(T_m^*, K_{1,n-1}) = m+n-3$ or $m+n-4$. Moreover, if $m+n-4 = (m-1)x+(m-2)y+2(m-3)z$ for some nonnegative integers x, y and z, then $r(T_m^*, K_{1,n-1}) = m+n-3$.*

Proof. By Proposition 6.2, $r(T_m^*, K_{1,n-1}) \leq m+n-3$. By Lemma 2.3 we have $r(T_m^*, K_{1,n-1}) \geq m+n-4$. If $m+n-4 = (m-1)x+(m-2)y+2(m-3)z$ for some nonnegative integers x, y and z, setting $G = xK_{m-1} \cup yK_{m-2} \cup zK_{m-3,m-3}$ we find $\Delta(\overline{G}) \leq n-2$. Clearly, G does not contain any copies of T_m^*, and $\overline{G}$ does not contain any copies of $K_{1,n-1}$. Thus, $r(T_m^*, K_{1,n-1}) > |V(G)| = m+n-4$ and so $r(T_m^*, K_{1,n-1}) = m+n-3$. This proves the theorem.

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