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Ramanujan's theta functions and sums of triangular numbers

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Abstract

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and the set of positive integers, respectively. For $a,b,c,n\in\mathbb{Z}^+$ let $N(a,b,c;n)$ be the number of representations of n by $ax^2+by^2+cz^2$, and let $t(a,b,c;n)$ be the number of representations of n by $ax(x+1)/2+by(y+1)/2+cz(z+1)/2$ $(x,y,z\in\mathbb{Z})$. In this paper, by using Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ we reveal some general relations between $t(a,b,c;n)$ and $N(a,b,c;8n+a+b+c)$.

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1. Introduction

Let $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{N}$ be the set of integers, the set of positive integers and the set of nonnegative integers, respectively. The numbers $x(x+1)/2$ $(x\in\mathbb{Z})$ are called triangular numbers. For $k,n\in\mathbb{Z}^+$ let $r_k(n)$ be the number of integral solutions to $n=x_1^2+\cdots+x_k^2$, and let $t_k(n)$ be the number of integral solutions to $n=\frac{x_1(x_1+1)}{2}+\cdots+\frac{x_k(x_k+1)}{2}$. In 1828 Jacobi showed that

$$r_4(n)=8\sum_{d\mid n,4\nmid d}d.$$

In 1801 Gauss (see [9, p.262]) proved that if $n>4$ is squarefree, then

$$r_3(n)=\begin{cases}24h(-n) & \text{if } n\equiv 3 \pmod 8,\\ 12h(-4n) & \text{if } n\equiv 1,2,5,6 \pmod 8,\\ 0 & \text{if } n\equiv 7 \pmod 8,\end{cases}$$

where $h(d)$ is the number of classes consisting of primitive binary quadratic forms of discriminant d. Suppose $n=2^{\alpha_0}\prod_{i=1}^s p_i^{\alpha_i}$, where $p_1,\ldots,p_s$ are distinct odd primes and

$\alpha_0, \alpha_1, \dots, \alpha_s \in \mathbb{N}$. In 1907 Hurwitz (see [9, p.271]) proved that

$$r_3(n^2) = 6\prod_{i=1}^{s}\Big(\frac{p_i^{\alpha_i+1}-1}{p_i-1} - (-1)^{\frac{p_i-1}{2}}\frac{p_i^{\alpha_i}-1}{p_i-1}\Big).$$

In 1998 Bateman and Knopp [3] showed that

$$t_k(n) = \frac{2}{2+\binom{k}{4}} r_k(8n+k) \quad \text{for } k \le 7.$$

Let $\mathbb{Z}^k = \underbrace{\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}}_{k\ times}$ and $\mathbb{N}^k = \underbrace{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}_{k\ times}$. For $a_1, a_2, \dots, a_k \in \mathbb{Z}^+$ $(k \ge 2)$ and $n \in \mathbb{N}$ set

$$N(a_1, a_2, \dots, a_k; n) = \big|\{(x_1, \dots, x_k) \in \mathbb{Z}^k \mid n = a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2\}\big|,$$

$$t(a_1, a_2, \dots, a_k; n)$$
$$= \Big|\Big\{(x_1, \dots, x_k) \in \mathbb{Z}^k \mid n = a_1\frac{x_1(x_1+1)}{2} + a_2\frac{x_2(x_2+1)}{2} + \cdots + a_k\frac{x_k(x_k+1)}{2}\Big\}\Big|.$$

Note that $\frac{x(x+1)}{2} = \frac{(-1-x)(-x)}{2}$. We see that

$$t(a_1, a_2, \dots, a_k; n)$$
$$= 2^k\Big|\Big\{(x_1, \dots, x_k) \in \mathbb{N}^k \mid n = a_1\frac{x_1(x_1+1)}{2} + a_2\frac{x_2(x_2+1)}{2} + \cdots + a_k\frac{x_k(x_k+1)}{2}\Big\}\Big|.$$

In 1862 Liouville ([9, p.23]) proved that for $a, b, c \in \mathbb{Z}^+$, $t(a, b, c; n) \ge 1$ for all $n \in \mathbb{Z}^+$ if and only if $(a, b, c) = (1, 1, 1), (1, 1, 2), (1, 1, 4), (1, 1, 5), (1, 2, 2), (1, 2, 3)$ or $(1, 2, 4)$. In 1924, Bell [4] gave transformation relations between $N(a, b, c; n)$ and $r_3(n)$ for $(a, b, c) = (1, 1, 2), (1, 1, 4), (1, 1, 8), (1, 2, 2), (1, 2, 4), (1, 2, 8), (1, 4, 4), (1, 4, 8), (1, 8, 8)$. Hürlimann[11] gave similar results for $(a, b, c) = (1, 2, 16), (1, 8, 16)$. For the formulas for $N(a, b, c; n^2)$ similar to Hurwitz's formula for $r_3(n^2)$ see [4,8,10-13,19].

Let $a, b, c, d, n \in \mathbb{Z}^+$. From 1859 to 1866 Liouville made about 90 conjectures on $N(a, b, c, d; n)$ in a series of papers. Most conjectures of Liouville have been proved. See Cooper's survey paper [7], Dickson's historical comments [9] and Williams' book [21]. In 2011, the author [15, Theorem 2.3] found two general relations between $t(a, b; n)$ and $N(a, b; 8n + a + b)$. Recently, the author and Wang (see [16,20]) revealed new connections between $t(a, b, c, d; n)$ and $N(a, b, c, d; 8n + a + b + c + d)$. They do not need assuming $a + b + c + d \le 8$. More recently Yao [18] and Xia and Zhong [17] confirmed some conjectures posed by the author in [16].

For $a_1, a_2, \dots, a_k \in \mathbb{Z}^+$ $(k \ge 2)$ define

$$C(a_1, \dots, a_k) = \binom{i_1}{4} + \binom{i_1}{2} i_2 + i_1 i_3,$$

where i_j denotes the number of elements in $\{a_1, \dots, a_k\}$ which are equal to j. In 2005 Adiga, Cooper and Han [1] showed that for $n \in \mathbb{N}$,

$$t(a_1, a_2, \dots, a_k; n)$$
$$= \frac{2}{2 + C(a_1, \dots, a_k)} N(a_1, \dots, a_k; 8n + a_1 + \cdots + a_k) \text{ for } a_1 + \cdots + a_k \le 7. \tag{1.1}$$

In 2008 Baruah, Cooper and Hirschhorn [2] proved that for $n \in \mathbb{N}$,

$$\begin{aligned} &t(a_1, a_2, \ldots, a_k; n) \\ (1.2) \qquad &= \frac{2}{2 + C(a_1, \ldots, a_k)}(N(a_1, \ldots, a_k; 8n+8) - N(a_1, \ldots, a_k; 2n+2)) \\ &\text{for } a_1 + \cdots + a_k = 8. \end{aligned}$$

In Section 2 we list some useful identities involving Ramanujan's theta functions. Let $m, n \in \mathbb{Z}^+$. In Section 3, using Ramanujan's theta functions we prove that

$$(1.3) \qquad t(1,1,8;n) = \begin{cases} \frac{1}{3}r_3(4n+5) + 2(-1)^{\frac{m+1}{2}}m & \text{if } 4n+5 = m^2 \text{ for some } m \in \mathbb{Z}^+, \\ \frac{1}{3}r_3(4n+5) & \text{otherwise.} \end{cases}$$

Let $m \equiv 1, 4, 5 \pmod 8$. Suppose that there is an odd prime divisor p of m such that $\left(\frac{4n+5}{p}\right) = -1$, where $(\frac{a}{p})$ is the Legendre symbol. Using (1.3) we deduce that

$$(1.4) \qquad t(1,1,8,m;n) = \frac{1}{2}N(1,1,8,m;8n+10+m),$$

which confirms [16, Conjectures 2.6 and 2.8]. We also show that for any $n \in \mathbb{Z}^+$,

$$(1.5) \qquad t(1,3,9;n) = \frac{1}{2}N(1,3,9;8n+13).$$

Let $a, b, n \in \mathbb{Z}^+$ with $2 \nmid a$. In Section 4, using Ramanujan's theta functions we prove that

$$(1.6) \qquad t(a,3a,2b;n) = \frac{2}{3}N(a,3a,2b;8n+4a+2b) \quad \text{for odd } b.$$

When b is even, similar results are given in Theorems 4.2 and 4.3. Such formulas are better than (1.1) and (1.2) since they provide infinite families of identities.

Let $a, c, n \in \mathbb{Z}^+$ with $2 \nmid a$. In Section 5 we obtain formulas for $t(a, 27a, c; n)$ and $t(3a, 25a, c; n)$ under certain conditions. For instance, if $c \equiv \pm 2 \pmod{12}$ and $n \equiv -c \pmod 3$, then

$$(1.7) \qquad t(a,27a,c;n) = \frac{2}{3}N(a,27a,c;8n+28a+c).$$

Let $a, b, c, n \in \mathbb{Z}^+$. In Section 6 we reveal three general relations between $t(a, b, c; n)$ and $N(a, b, c; 8n+a+b+c)$, and show that $t(1, 1, 9; n) \geq 1$ if and only if $n \not\equiv 5, 8 \pmod 9$. Based on calculations with Maple we pose four challenging conjectures.

2. Ramanujan's theta functions

Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \ (|q| < 1).$$

For $a_1, \ldots, a_k \in \mathbb{Z}^+$ and $|q| < 1$, it is easy to see that

$$\sum_{n=0}^{\infty} N(a_1, \ldots, a_k; n) q^n = \varphi(q^{a_1}) \cdots \varphi(q^{a_k}), \tag{2.1}$$

$$\sum_{n=0}^{\infty} t(a_1, \ldots, a_k; n) q^n = 2^k \psi(q^{a_1}) \cdots \psi(q^{a_k}). \tag{2.2}$$

There are many identities involving $\varphi(q)$ and $\psi(q)$. Suppose $|q| < 1$. From [2, Lemma 4.1] or [5] we know that

$$\psi(q)^2 = \varphi(q)\psi(q^2), \tag{2.3}$$

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{2.4}$$

$$\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2, \tag{2.5}$$

$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}). \tag{2.6}$$

By (2.4),

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8) = \varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8). \tag{2.7}$$

By [16, Lemma 2.4],

$$\varphi(q)^2 = \varphi(q^8)^2 + 4q^4\psi(q^{16})^2 + 4q^2\psi(q^8)^2 + 4q\varphi(q^{16})\psi(q^8) + 8q^5\psi(q^8)\psi(q^{32}). \tag{2.8}$$

By [16, Lemma 2.3],

$$\begin{aligned}\varphi(q)\varphi(q^3) = \varphi(q^{16})\varphi(q^{48}) + 4q^{16}\psi(q^{32})\psi(q^{96}) + 2q\varphi(q^{48})\psi(q^8) + 2q^3\varphi(q^{16})\psi(q^{24}) \\ + 6q^4\psi(q^8)\psi(q^{24}) + 4q^{13}\psi(q^8)\psi(q^{96}) + 4q^7\psi(q^{24})\psi(q^{32}).\end{aligned} \tag{2.9}$$

It is also known that (see [14, pp.113-114] and [6, p.71])

$$\psi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{1-q^n} \quad \text{and} \quad \varphi(-q) = \varphi(q) - 4q\psi(q^8) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{1-q^{2n}}. \tag{2.10}$$

Using theta function identities we may establish some relations between $t(a, b, c; n)$ and $N(a, b, c; n)$, where $a, b, c, n \in \mathbb{Z}^+$. As two examples, for later use we deduce the relation between $t(1, 1, 2; n)$ and $r_3(n)$, and the relation between $N(1, 1, 8; n)$ and $r_3(n)$. By (2.1), (2.3) and (2.4), for $|q| < 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} r_3(n)q^n &= \varphi(q)^3 = (\varphi(q^4) + 2q\psi(q^8))^3 \\ &= \varphi(q^4)^3 + 6q\varphi(q^4)^2\psi(q^8) + 12q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3 \\ &= \varphi(q^4)^3 + 6q\varphi(q^4)\psi(q^4)^2 + 12q^2\psi(q^4)^2\psi(q^8) + 8q^3\psi(q^8)^3.\end{aligned} \tag{2.11}$$

Collecting the terms of the form q^{4n+2} in (2.11) yields

$$\sum_{n=0}^{\infty} r_3(4n+2)q^{4n+2} = 12q^2\psi(q^4)^2\psi(q^8)$$

and so

$$\sum_{n=0}^{\infty} r_3(4n+2)q^n = 12\psi(q)^2\psi(q^2) = \frac{3}{2}\sum_{n=0}^{\infty} t(1,1,2;n)q^n.$$

Hence

$$t(1,1,2;n) = \frac{2}{3}r_3(4n+2). \tag{2.12}$$

By (2.11), we also have

$$\sum_{n=0}^{\infty} r_3(4n+1)q^{4n+1} = 6q\varphi(q^4)\psi(q^4)^2 \text{ and so } \sum_{n=0}^{\infty} r_3(4n+1)q^n = 6\varphi(q)\psi(q)^2. \tag{2.13}$$

Using (2.8) we see that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,1,8;n)q^n = \varphi(q)^2\varphi(q^8)\\&= \big(\varphi(q^8)^2 + 4q^4\psi(q^{16})^2 + 4q^2\psi(q^8)^2 + 4q\varphi(q^{16})\psi(q^8) + 8q^5\psi(q^8)\psi(q^{32})\big)\varphi(q^8).\end{aligned}$$

Collecting the terms of the form q^{8n+2} in the above expansion yields

$$\sum_{n=0}^{\infty} N(1,1,8;8n+2)q^{8n+2} = 4q^2\psi(q^8)^2\varphi(q^8)$$

and hence

$$\sum_{n=0}^{\infty} N(1,1,8;8n+2)q^n = 4\varphi(q)\psi(q)^2. \tag{2.14}$$

Comparing (2.14) with (2.13) yields

$$N(1,1,8;8n+2) = \frac{2}{3}r_3(4n+1), \tag{2.15}$$

which was first obtained by Bell [4].

3. Formulas for $t(1,1,8;n)$ and $t(1,3,9;n)$

Based on calculations on Maple, in this section we present the relation between $t(1,1,8;n)$ and $N(1,1,8;8n+10)$, and the relation between $t(1,3,9;n)$ and $N(1,3,9;8n+13)$.

Theorem 3.1. *For $n \in \mathbb{N}$ we have*

$$\begin{aligned}&t(1,1,8;n) - \frac{1}{2}N(1,1,8;8n+10)\\&= t(1,1,8;n) - \frac{1}{3}r_3(4n+5) = \begin{cases} 2(-1)^{\frac{m+1}{2}}m & \text{if } 4n+5 = m^2 \text{ for some } m \in \mathbb{Z}^+,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. By (2.15), $N(1,1,8;8n+10) = \frac{2}{3}r_3(4n+5)$. Set $s(n) = t(1,1,8;n) - \frac{1}{3}r_3(4n+5)$. By (2.2),(2.10),(2.13) and the fact $r_3(1) = 6$, for $0 < |q| < 1$ we have

$$\sum_{n=0}^{\infty} s(n)q^n = \sum_{n=0}^{\infty} t(1,1,8;n)q^n - \frac{1}{3}\sum_{n=0}^{\infty} r_3(4n+5)q^n$$

$$\begin{aligned}&= 8\psi(q)^2\psi(q^8) - \frac{1}{3q}\sum_{n=1}^{\infty} r_3(4n+1)q^n \\ &= 8\psi(q)^2\psi(q^8) - \frac{1}{3q}\big(6\psi(q)^2\varphi(q) - r_3(1)\big) \\ &= \frac{2\psi(q)^2(4q\psi(q^8) - \varphi(q)) + 2}{q} = 2\frac{1-\varphi(-q)\psi(q)^2}{q}.\end{aligned}$$

Thus, appealing to (2.10) and Jacobi's identity (see [14, p.8])

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}} \quad (|q|<1) \tag{3.1}$$

we get

$$\begin{aligned}\sum_{n=0}^{\infty} s(n)q^n &= 2\frac{1-\varphi(-q)\psi(q)^2}{q} = \frac{2}{q}\Big(1 - \prod_{n=1}^{\infty}\frac{(1-q^n)^2}{1-q^{2n}} \cdot \prod_{n=1}^{\infty}\frac{(1-q^{2n})^4}{(1-q^n)^2}\Big) \\ &= \frac{2}{q}\big(1 - \prod_{n=1}^{\infty}(1-q^{2n})^3\big) = \frac{2}{q}\Big(1 - \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)}\Big) \\ &= 2\sum_{k=1}^{\infty}(-1)^{k+1}(2k+1)q^{k^2+k-1} = 2\sum_{k=1}^{\infty}(-1)^{k+1}(2k+1)q^{\frac{(2k+1)^2-5}{4}}.\end{aligned}$$

Now comparing the coefficients of q^n on both sides yields

$$s(n) = \begin{cases} 2(-1)^{\frac{m+1}{2}}m & \text{if } 4n+5 = m^2 \text{ for some } m \in \mathbb{N}, \\ 0 & \text{otherwise.}\end{cases}$$

This proves the theorem.

Corollary 3.1. *Suppose $n \in \mathbb{Z}^+$. If $n \equiv 0 \pmod 2$, $n \equiv 0 \pmod 3$, $n \equiv 2, 3 \pmod 5$ or $n \equiv 0, 2, 3 \pmod 7$, then*

$$t(1,1,8;n) = \frac{1}{2}N(1,1,8;8n+10).$$

Proof. If $2 \mid n$, then $4n+5 \equiv 5 \pmod 8$. If $3 \mid n$, then $4n+5 \equiv 2 \pmod 3$. If $n \equiv 2, 3 \pmod 5$, then $4n+5 \equiv 2, 3 \pmod 5$. If $n \equiv 0, 2, 3 \pmod 7$, then $4n+5 \equiv 3, 5, 6 \pmod 7$. Thus, if n satisfies one of the assumed conditions, then $4n+5$ is not a square and so $t(1,1,8;n) = \frac{1}{2}N(1,1,8;8n+10)$ by Theorem 3.1.

Theorem 3.2. *Let $m, n \in \mathbb{Z}^+$ with $m \equiv 1 \pmod 4$ or $m \equiv 4 \pmod 8$. Suppose that there is an odd prime divisor p of m such that $\big(\frac{4n+5}{p}\big) = -1$. Then*

$$t(1,1,8,m;n) = \frac{1}{2}N(1,1,8,m;8n+10+m).$$

Proof. Suppose that p is an odd prime divisor of m with $\big(\frac{4n+5}{p}\big) = -1$. For $w \in \mathbb{Z}$ we see that

$$\Big(\frac{4(n - m\frac{w(w-1)}{2}) + 5}{p}\Big) = \Big(\frac{4n+5}{p}\Big) = -1.$$

Hence $4(n - m\frac{w(w-1)}{2}) + 5$ is not a square. Now, from Theorem 3.1 we derive that

$$\begin{aligned} t(1,1,8,m;n) &= \sum_{w\in\mathbb{Z}} t\big(1,1,8;n-mw(w+1)/2\big) \\ &= \frac{1}{2}\sum_{w\in\mathbb{Z}} N(1,1,8;8n+10-m\cdot 4w(w+1)) \\ &= \frac{1}{2}\sum_{w\in\mathbb{Z}} N(1,1,8;8n+10+m-m(2w+1)^2). \end{aligned}$$

Since $a^2 \equiv 0,1 \pmod 4$ and $a^2 \equiv 0,1,4 \pmod 8$ for any $a \in \mathbb{Z}$, we see that $x^2+y^2 \not\equiv 3 \pmod 4$ and $x^2+y^2 \not\equiv 6 \pmod 8$ for any $x,y \in \mathbb{Z}$. If $m \equiv 1 \pmod 4$ and $8n+10+m-m(2w)^2 = x^2+y^2+8z^2$ for some $x,y,z,w \in \mathbb{Z}$, then $x^2+y^2 \equiv 10+m \equiv 3 \pmod 4$. This is impossible. If $m \equiv 4 \pmod 8$ and $8n+10+m-m(2w)^2 = x^2+y^2+8z^2$ for some $x,y,z,w \in \mathbb{Z}$, then $x^2+y^2 \equiv 10+m \equiv 6 \pmod 8$. This is also impossible. Hence, for $m \equiv 1 \pmod 4$ or $m \equiv 4 \pmod 8$,

$$\begin{aligned} t(1,1,8,m;n) &= \frac{1}{2}\sum_{w\in\mathbb{Z}} N(1,1,8;8n+10+m-m(2w+1)^2) \\ &= \frac{1}{2}\sum_{w\in\mathbb{Z}} N(1,1,8;8n+10+m-mw^2) \\ &= \frac{1}{2}N(1,1,8,m;8n+10+m). \end{aligned}$$

This proves the theorem.

Corollary 3.2 ([16, Conjectures 2.6 and 2.8]). *Let $n \in \mathbb{Z}^+$. Then*

$$t(1,1,5,8;n) = \frac{1}{2}N(1,1,5,8;8n+15) \quad \textit{for} \quad n \equiv 2,3 \pmod 5$$

and

$$t(1,1,8,13;n) = \frac{1}{2}N(1,1,8,13;8n+23) \quad \textit{for} \quad n \equiv 0,4,7,8,9,10 \pmod{13}.$$

Proof. If $n \equiv 2,3 \pmod 5$, then $\big(\frac{4n+5}{5}\big) = -1$. If $n \equiv 0,4,7,8,9,10 \pmod{13}$, then $\big(\frac{4n+5}{13}\big) = -1$. Now putting $m = 5,13$ in Theorem 3.2 yields the result.

Remark 3.1 By Theorem 3.2, for $n \equiv 0 \pmod 3$ we have $(\frac{4n+5}{3}) = -1$ and so $t(1,1,8,9;n) = \frac{1}{2}N(1,1,8,9;8n+19)$ and $t(1,1,8,12;n) = \frac{1}{2}N(1,1,8,12;8n+22)$, which were conjectured by the author in [16, Conjectures 2.2 and 2.7] and first confirmed by Yao in [18].

For $a,b,c,n \in \mathbb{Z}^+$ it is clear that

$$\begin{aligned} &n = a\frac{x(x+1)}{2} + b\frac{y(y+1)}{2} + c\frac{z(z+1)}{2} \\ \iff\ &8n+a+b+c = a(2x+1)^2 + b(2y+1)^2 + c(2z+1)^2. \end{aligned}$$

Thus,

$$t(a,b,c;n) = \big|\{(x,y,z)\in\mathbb{Z}^3 \mid 8n+a+b+c = ax^2+by^2+cz^2,\ 2\nmid xyz\}\big|. \tag{3.2}$$

Theorem 3.3. *For $n\in\mathbb{Z}^+$ we have*

$$t(1,3,9;n) = \frac{1}{2}N(1,3,9;8n+13).$$

Proof. By (2.1), (2.7) and (2.9),

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,3,9;n)q^n &= \varphi(q)\varphi(q^3)\varphi(q^9)\\ &= \big(\varphi(q^{16})\varphi(q^{48}) + 4q^{16}\psi(q^{32})\psi(q^{96}) + 2q\varphi(q^{48})\psi(q^8) + 2q^3\varphi(q^{16})\psi(q^{24})\\ &\quad + 6q^4\psi(q^8)\psi(q^{24}) + 4q^{13}\psi(q^8)\psi(q^{96}) + 4q^7\psi(q^{24})\psi(q^{32})\big)\\ &\quad \times \big(\varphi(q^{144}) + 2q^{36}\psi(q^{288}) + 2q^9\psi(q^{72})\big).\end{aligned} \tag{3.3}$$

Collecting the terms of the form q^{8n+13} in (3.3) and then applying (2.2) and (2.6) we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,3,9;8n+13)q^{8n+13}\\ &= 4q^{13}\psi(q^8)\psi(q^{96})\cdot\varphi(q^{144}) + 2q\varphi(q^{48})\psi(q^8)\cdot 2q^{36}\psi(q^{288}) + 6q^4\psi(q^8)\psi(q^{24})\cdot 2q^9\psi(q^{72})\\ &= 4q^{13}\psi(q^8)\big(\varphi(q^{144})\psi(q^{96}) + q^{24}\varphi(q^{48})\psi(q^{288})\big) + 12q^{13}\psi(q^8)\psi(q^{24})\psi(q^{72})\\ &= 16q^{13}\psi(q^8)\psi(q^{24})\psi(q^{72}) = 2q^{13}\sum_{n=0}^{\infty} t(1,3,9;n)q^{8n}.\end{aligned}$$

Now comparing the coefficients of q^{8n+13} on both sides yields

$$N(1,3,9;8n+13) = 2t(1,3,9;n).$$

Remark 3.2 One can similarly prove that

$$t(1,1,3;n) = \frac{1}{2}N(1,1,3;8n+5) \quad\text{and}\quad t(1,3,3;n) = \frac{1}{2}N(1,3,3;8n+7),$$

which can be deduced from (1.1).

4. Formulas for $t(a,3a,2b;n)$

For $a,b,n\in\mathbb{Z}^+$ with $2\nmid a$, in this section we establish general formulas for $t(a,3a,2b;n)$, which yield infinite families of identities.

By (2.1), (2.7) and (2.9), for $a,b\in\mathbb{Z}^+$ with $2\nmid a$ and $|q|<1$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(a,3a,2b;n)q^n &= \varphi(q^a)\varphi(q^{3a})\varphi(q^{2b})\\ &= \big(\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}) + 2q^a\varphi(q^{48a})\psi(q^{8a})\\ &\quad + 2q^{3a}\varphi(q^{16a})\psi(q^{24a}) + 6q^{4a}\psi(q^{8a})\psi(q^{24a}) + 4q^{13a}\psi(q^{8a})\psi(q^{96a})\\ &\quad + 4q^{7a}\psi(q^{24a})\psi(q^{32a})\big)\big(\varphi(q^{8b}) + 2q^{2b}\psi(q^{16b})\big).\end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $a, b \in \{1, 3, 5, \ldots\}$. For $n \in \mathbb{Z}^+$ we have*

$$t(a, 3a, 2b; n) = \frac{2}{3} N(a, 3a, 2b; 8n + 4a + 2b).$$

Proof. Since $4a + 2b \equiv 2 \pmod 4$, collecting the terms of the form $q^{8n+4a+2b}$ in (4.1) yields

$$\sum_{n=0}^{\infty} N(a, 3a, 2b; 8n + 4a + 2b) q^{8n+4a+2b} = 6q^{4a} \psi(q^{8a}) \psi(q^{24a}) \cdot 2q^{2b} \psi(q^{16b}).$$

Replacing q with $q^{1/8}$ gives

$$\begin{aligned}&\sum_{n=0}^{\infty} N(a, 3a, 2b; 8n + 4a + 2b) q^n \\ &= 12\psi(q^a)\psi(q^{3a})\psi(q^{2b}) = \frac{12}{8} \sum_{n=0}^{\infty} t(a, 3a, 2b; n) q^n.\end{aligned}$$

Now comparing the coefficients of q^n on both sides yields the result.

Theorem 4.2. *Let $a \in \{1, 3, 5, \ldots\}$ and $m \in \mathbb{Z}^+$. For $n \in \mathbb{Z}^+$ we have*

$$t(a, 3a, 8m; n) = \frac{2}{3} N(a, 3a, 8m; 8n + 4a + 8m) - 2N(a, 3a, 8m; 2n + a + 2m).$$

Proof. Set $b = 4m$. Collecting the terms of the form q^{8n+4a} in (4.1) we deduce that

$$\sum_{n=0}^{\infty} N(a, 3a, 2b; 8n + 4a) q^{8n+4a} = 6q^{4a} \psi(q^{8a}) \psi(q^{24a}) (\varphi(q^{8b}) + 2q^{2b} \psi(q^{16b})).$$

Replacing q with $q^{1/8}$ we obtain

$$\sum_{n=0}^{\infty} N(a, 3a, 8m; 8n + 4a) q^n = 6\psi(q^a)\psi(q^{3a})(\varphi(q^{4m}) + 2q^m \psi(q^{8m})). \tag{4.2}$$

On the other hand, using (2.1) and (2.4) we see that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(a, 3a, 8m; n) q^n \\ &= \varphi(q^a)\varphi(q^{3a})\varphi(q^{8m}) = (\varphi(q^{4a}) + 2q^a \psi(q^{8a}))(\varphi(q^{12a}) + 2q^{3a} \psi(q^{24a}))\varphi(q^{8m}).\end{aligned}$$

Collecting the terms of the form q^{2n+a} and then applying (2.6) we get

$$\begin{aligned}&\sum_{n=0}^{\infty} N(a, 3a, 8m; 2n + a) q^{2n+a} \\ &= (2q^a \psi(q^{8a})\varphi(q^{12a}) + 2q^{3a} \psi(q^{24a})\varphi(q^{4a}))\varphi(q^{8m}) \\ &= 2q^a \psi(q^{2a})\psi(q^{6a})\varphi(q^{8m})\end{aligned}$$

and so

$$\sum_{n=0}^{\infty} N(a,3a,8m;2n+a)q^n = 2\psi(q^a)\psi(q^{3a})\varphi(q^{4m}).$$

This together with (4.2) yields

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(a,3a,8m;8n+4a)-3N(a,3a,8m;2n+a))q^n\\ &= 12q^m\psi(q^a)\psi(q^{3a})\psi(q^{8m}) = \frac{12}{8}q^m\sum_{n=0}^{\infty} t(a,3a,8m;n)q^n.\end{aligned}$$

Now comparing the coefficients of q^{m+n} on both sides gives the result.

Theorem 4.3. *Let $a \in \{1,3,5,\dots\}$ and $m \in \mathbb{N}$. For $n \in \mathbb{Z}^+$ we have*

$$\begin{aligned}&t(a,3a,8m+4;n)\\ &= \begin{cases} \frac{2}{3}N(a,3a,8m+4;8n+4a+8m+4) & \text{if } n \equiv \frac{a-1}{2}+m \pmod 2,\\ \frac{2}{3}\big(N(a,3a,8m+4;8n+4a+8m+4) - N(a,3a,8m+4;2n+a+2m+1)\big) & \\ & \text{if } n \not\equiv \frac{a-1}{2}+m \pmod 2.\end{cases}\end{aligned}$$

Proof. Set $b = 4m+2$. Collecting the terms of the form q^{8n} in (4.1) we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(a,3a,2b;8n)q^{8n}\\ &= \big(\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a})\big)\varphi(q^{8b}) + 6q^{4a}\psi(q^{8a})\psi(q^{24a})\cdot 2q^{2b}\psi(q^{16b}).\end{aligned}$$

Replacing q with $q^{1/8}$ we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} N(a,3a,8m+4;8n)q^n &= \big(\varphi(q^{2a})\varphi(q^{6a}) + 4q^{2a}\psi(q^{4a})\psi(q^{12a})\big)\varphi(q^{4m+2})\\ &\quad + 12q^{m+(a+1)/2}\psi(q^a)\psi(q^{3a})\psi(q^{8m+4}).\end{aligned}\tag{4.3}$$

On the other hand, using (2.4) we see that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(a,3a,8m+4;n)q^n\\ &= \varphi(q^a)\varphi(q^{3a})\varphi(q^{8m+4}) = (\varphi(q^{4a})+2q^a\psi(q^{8a}))(\varphi(q^{12a})+2q^{3a}\psi(q^{24a}))\varphi(q^{8m+4}).\end{aligned}$$

Collecting the even powers of q we get

$$\sum_{n=0}^{\infty} N(a,3a,8m+4;2n)q^{2n} = (\varphi(q^{4a})\varphi(q^{12a}) + 4q^{4a}\psi(q^{8a})\psi(q^{24a}))\varphi(q^{8m+4})$$

and so

$$\sum_{n=0}^{\infty} N(a,3a,8m+4;2n)q^n = (\varphi(q^{2a})\varphi(q^{6a}) + 4q^{2a}\psi(q^{4a})\psi(q^{12a}))\varphi(q^{4m+2}).$$

This together with (4.3) yields

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(a,3a,8m+4;8n)-N(a,3a,8m+4;2n))q^n\\&=12q^{m+(a+1)/2}\psi(q^a)\psi(q^{3a})\psi(q^{8m+4})\\&=\frac{3}{2}q^{m+(a+1)/2}\sum_{n=0}^{\infty}t(a,3a,8m+4;n)q^n.\end{aligned}$$

Comparing the coefficients of $q^{m+(a+1)/2+n}$ on both sides yields

$$\begin{aligned}&t(a,3a,8m+4;n)\\&=\frac{2}{3}\big(N(a,3a,8m+4;8n+4a+8m+4)-N(a,3a,8m+4;2n+a+2m+1)\big).\end{aligned}$$

Now assume $n\equiv\frac{a-1}{2}+m \pmod 2$. Then $2n+2m+a+1\equiv a-1+2m+2m+a+1\equiv 2a \pmod 4$. If $2n+2m+a+1=ax^2+3ay^2+(8m+4)z^2$ for some $x,y,z\in\mathbb{Z}$, we must have $a(x^2+3y^2)\equiv 2n+2m+a+1\equiv 2a \pmod 4$ and so $x^2+3y^2\equiv 2 \pmod 4$. If $2\mid x-y$, then $4\mid x^2+3y^2$. If $2\nmid x-y$, then x^2+3y^2 is odd. Thus, $x^2+3y^2\not\equiv 2 \pmod 4$ and we get a contradiction. Therefore $N(a,3a,8m+4;2n+a+2m+1)=0$. This completes the proof.

5. Formulas for $t(a,27a,c;n)$ and $t(3a,25a,c;n)$

The purpose of this section is to present some formulas for $t(a,27a,c;n)$ and $t(3a,25a,c;n)$ by using Theorems 4.1-4.3, where $a,c,n\in\mathbb{Z}^+$ and $2\nmid a$. For later convenience we define $t(a,b,c;m)=0$ for $a,b,c\in\mathbb{Z}^+$ and $m\notin\mathbb{N}$. Now let us begin with two lemmas.

Lemma 5.1. *Let $a,b,c,n\in\mathbb{Z}^+$ with $3\nmid a$ and $n\equiv a \pmod 3$. Then*

$$t(a,3b,3c;n)=t\big(3a,b,c;\frac{n-a}{3}\big).$$

Proof. If $8n+a+3b+3c=ax^2+3by^2+3cz^2$ for some odd integers x,y and z, then clearly $3\mid x$ and so $8n+a+3b+3c=a(3x)^2+3by^2+3cz^2$ for some odd integers x,y and z. That is, $\frac{8n+a+3b+3c}{3}=3ax^2+by^2+cz^2$ for some odd integers x,y and z. Thus, applying (3.2) we obtain

$$\begin{aligned}&t(a,3b,3c;n)\\&=\big|\{(x,y,z)\in\mathbb{Z}^3 \mid 8n+a+3b+3c=ax^2+3by^2+3cz^2,\ 2\nmid xyz\}\big|\\&=\big|\{(x,y,z)\in\mathbb{Z}^3 \mid 8n+a+3b+3c=a(3x)^2+3by^2+3cz^2,\ 2\nmid xyz\}\big|\\&=\big|\{(x,y,z)\in\mathbb{Z}^3 \mid \frac{8n+a+3b+3c}{3}=3ax^2+by^2+cz^2,\ 2\nmid xyz\}\big|\\&=t\big(3a,b,c;\frac{n-a}{3}\big).\end{aligned}$$

This proves the lemma.

Lemma 5.2. *Let $a, b, c, n \in \mathbb{Z}^+$ with $3 \nmid a$, $a \equiv b \pmod 3$ and $n \equiv 2a \pmod 3$. Then*

$$t(a, b, 9c; n) = \begin{cases} t\big(a, b, c; \dfrac{n-a-b}{9}\big) & \text{if } n \equiv a+b \pmod 9, \\ 0 & \text{otherwose.} \end{cases}$$

Proof. Since $n \equiv a+b \pmod 3$ we see that $8n+a+b+9c \equiv 0 \pmod 3$. If $8n+a+b+9c = ax^2+by^2+9cz^2$ for some $x, y, z \in \mathbb{Z}$, we must have $x^2+y^2 \equiv 0 \pmod 3$ and so $x \equiv y \equiv 0 \pmod 3$. Thus $8n+a+b+9c = a(3x)^2+b(3y)^2+9cz^2$ for some $x, y, z \in \mathbb{Z}$. This implies $9 \mid 8n+a+b$. Hence, applying (3.2) we see that

$$\begin{aligned} & t(a, b, 9c; n) \\ &= \big|\big\{(x, y, z) \in \mathbb{Z}^3 \bigm| 8n+a+b+9c = ax^2+by^2+9cz^2,\ 2 \nmid xyz\big\}\big| \\ &= \big|\big\{(x, y, z) \in \mathbb{Z}^3 \bigm| 8n+a+b+9c = 9ax^2+9by^2+9cz^2,\ 2 \nmid xyz\big\}\big| \\ &= \big|\big\{(x, y, z) \in \mathbb{Z}^3 \bigm| \frac{8n+a+b+9c}{9} = ax^2+by^2+cz^2,\ 2 \nmid xyz\big\}\big| \\ &= \big|\big\{(x, y, z) \in \mathbb{Z}^3 \bigm| 8\frac{n-a-b}{9}+a+b+c = ax^2+by^2+cz^2,\ 2 \nmid xyz\big\}\big| \\ &= \begin{cases} t\big(a, b, c; \dfrac{n-a-b}{9}\big) & \text{if } n \equiv a+b \pmod 9, \\ 0 & \text{otherwose.} \end{cases} \end{aligned}$$

This proves the lemma.

Theorem 5.1. *Let $n \in \mathbb{Z}^+$.*
(i) *If $n \equiv 0 \pmod 3$, then $t(1, 2, 9; n) = \frac{2}{3}N(1, 2, 9; 8n+12)$.*
(ii) *If $n \equiv 1 \pmod 3$, then*

$$\begin{aligned} t(1, 3, 27; n) &= \frac{1}{2}N(1, 3, 27; 8n+31), \\ t(1, 9, 27; n) &= \frac{1}{2}N(1, 9, 27; 8n+37), \\ t(1, 27, 27; n) &= \frac{1}{2}N(1, 27, 27; 8n+55), \\ t(1, 9, 18; n) &= \frac{2}{3}N(1, 9, 18; 8n+28). \end{aligned}$$

(iii) *If $n \equiv 2 \pmod 3$, then*

$$\begin{aligned} t(1, 1, 18; n) &= \frac{2}{3}N(1, 1, 18; 8n+20), \\ t(2, 9, 9; n) &= \frac{2}{3}N(2, 9, 9; 8n+20), \\ t(1, 1, 27; n) &= \frac{1}{2}N(1, 1, 27; 8n+29). \end{aligned}$$

Proof. We first prove (i). Suppose $3 \mid n$. For $x \in \mathbb{Z}$ we have $x^2 \equiv 0, 1 \pmod 3$. Thus, if $8n+12 = x^2+2y^2+z^2$ for $x, y, z \in \mathbb{Z}$, we must have $3 \mid x$ or $3 \mid z$. Hence $8n+12 = x^2+2y^2+(3z)^2$ for some $x, y, z \in \mathbb{Z}$. Applying (1.1), (3.2) and the above we deduce that

$$t(1, 2, 9; n) = \big|\{(x, y, z) \in \mathbb{Z}^3 \mid x^2+2y^2+(3z)^2 = 8n+12,\ 2 \nmid xyz\}\big|$$

$$
\begin{aligned}
&= \left|\{(x,y,z)\in\mathbb{Z}^3 \mid x^2+2y^2+z^2=8n+12,\ 2\nmid xyz\}\right| \\
&= t(1,1,2;n+1) = \frac{2}{3}N(1,1,2;8(n+1)+4) \\
&= \frac{2}{3}\left|\{(x,y,z)\in\mathbb{Z}^3 \mid x^2+2y^2+z^2=8n+12\}\right| \\
&= \frac{2}{3}\left|\{(x,y,z)\in\mathbb{Z}^3 \mid x^2+2y^2+(3z)^2=8n+12\}\right| \\
&= \frac{2}{3}N(1,2,9;8n+12).
\end{aligned}
$$

Now we consider (ii). Suppose $n=3m+1$. By Lemma 5.1 and Theorem 3.3,

$$
\begin{aligned}
t(1,3,27;n) = t(3,1,9;m) &= \frac{1}{2}N(1,3,9;8m+13) \\
&= \frac{1}{2}\left|\{(x,y,z)\in\mathbb{Z}^3 \mid x^2+3y^2+9z^2=8m+13\}\right| \\
&= \frac{1}{2}\left|\{(x,y,z)\in\mathbb{Z}^3 \mid 3x^2+(3y)^2+27z^2=24m+39\}\right| \\
&= \frac{1}{2}\left|\{(x,y,z)\in\mathbb{Z}^3 \mid 3x^2+y^2+27z^2=24m+39\}\right| \\
&= \frac{1}{2}N(1,3,27;24m+39) = \frac{1}{2}N(1,3,27;8n+31).
\end{aligned}
$$

The remaining results in part (ii) can be proved similarly.

Finally we consider (iii). Suppose $n=3m+2$. If $x^2+y^2+18z^2=24m+36$ for $x,y,z\in\mathbb{Z}$, then $x^2+y^2\equiv 0 \pmod 3$. This yields $x\equiv y\equiv 0 \pmod 3$ and so $3\mid m$. Thus, when $3\nmid m$ we have $N(1,1,18;8n+20)=N(1,1,18;24m+36)=0$ and so $t(1,1,18;n)=0$ by (3.2). Now assume $3\mid m$. Using Lemma 5.2 and (1.1) we see that

$$
\begin{aligned}
t(1,1,18;n) = t(1,1,2;m/3) &= \frac{2}{3}N(1,1,2;8m/3+4) \\
&= \frac{2}{3}\left|\{(x,y,z)\in\mathbb{Z}^3 \mid x^2+y^2+2z^2=8m/3+4\}\right| \\
&= \frac{2}{3}\left|\{(x,y,z)\in\mathbb{Z}^3 \mid (3x)^2+(3y)^2+18z^2=24m+36\}\right| \\
&= \frac{2}{3}\left|\{(x,y,z)\in\mathbb{Z}^3 \mid x^2+y^2+18z^2=8(3m+2)+20\}\right| \\
&= \frac{2}{3}N(1,1,18;8n+20).
\end{aligned}
$$

The remaining results in part (iii) can be proved similarly.

Lemma 5.3. *Let* $a,b,c\in\mathbb{Z}^+$ *with* $3\nmid c$.
(i) *If* $m\in\mathbb{Z}^+$ *and* $m\equiv a-c \pmod 3$, *then* $N(a,3b,c;m)=N(9a,3b,c;m)$.
(ii) *If* $n\in\mathbb{Z}^+$ *and* $n\equiv -c \pmod 3$, *then* $t(a,3b,c;n)=t(9a,3b,c;n-a)$.

Proof. We first prove (i). Suppose $m\in\mathbb{Z}^+$, $m\equiv a-c \pmod 3$ and $m=ax^2+3by^2+cz^2$ for $x,y,z\in\mathbb{Z}$. If $3\nmid x$, then $m=ax^2+3by^2+cz^2\equiv a+cz^2\equiv a+c$ or $a \pmod 3$. But $m\equiv a-c\not\equiv a+c,a \pmod 3$. This is a contradiction. Thus $3\mid x$ and so $m=9ax^2+3by^2+cz^2$ for some $x,y,z\in\mathbb{Z}$. Hence

$$N(a,3b,c;m) = \left|\{(x,y,z)\in\mathbb{Z}^3 \mid ax^2+3by^2+cz^2=m\}\right|$$

$$= \left|\{(x,y,z)\in\mathbb{Z}^3 \mid 9ax^2+3by^2+cz^2=m\}\right|$$
$$= N(9a,3b,c;m).$$

This proves (i).

Now we consider (ii). Suppose $n\in\mathbb{Z}^+$ and $n\equiv -c \pmod 3$. Then $8n+a+3b+c\equiv a-c \pmod 3$. Thus, applying (3.2) and the proof of (i) we deduce that

$$\begin{aligned} t(a,3b,c;n) &= \left|\{(x,y,z)\in\mathbb{Z}^3 \mid ax^2+3by^2+cz^2=8n+a+3b+c,\ 2\nmid xyz\}\right| \\ &= \left|\{(x,y,z)\in\mathbb{Z}^3 \mid 9ax^2+3by^2+cz^2=8n+a+3b+c,\ 2\nmid xyz\}\right| \\ &= \left|\{(x,y,z)\in\mathbb{Z}^3 \mid 9ax^2+3by^2+cz^2=8(n-a)+9a+3b+c,\ 2\nmid xyz\}\right| \\ &= t(9a,3b,c;n-a), \end{aligned}$$

which proves the lemma.

Theorem 5.2. *Let $a,c\in\mathbb{Z}^+$ with $2\nmid a$ and $c\equiv 2,4 \pmod 6$. For $n\in\mathbb{Z}^+$ with $n\equiv -c \pmod 3$ we have*

$$t(a,27a,c;n) = \begin{cases} \frac{2}{3}N(a,27a,c;8n+28a+c) & \text{if } c\equiv 2,10 \pmod{12},\\ \frac{2}{3}\big(N(a,27a,c;8n+28a+c)-N(a,27a,c;2n+7a+\frac{c}{4})\big) & \text{if } c\equiv 4,20 \pmod{24},\\ \frac{2}{3}N(a,27a,c;8n+28a+c)-2N(a,27a,c;2n+7a+\frac{c}{4}) & \text{if } c\equiv 8,16 \pmod{24}. \end{cases}$$

Proof. By Lemma 5.3(ii) and Theorems 4.1-4.3,

$$\begin{aligned} t(a,27a,c;n) &= t(9a,27a,c;n-a) \\ &= \begin{cases} \frac{2}{3}N(9a,27a,c;8(n-a)+36a+c) & \text{if } c\equiv 2,10 \pmod{12},\\ \frac{2}{3}\big(N(9a,27a,c;8(n-a)+36a+c)-N(9a,27a,c;2(n-a)+9a+\frac{c}{4})\big) \\ \qquad \text{if } c\equiv 4,20 \pmod{24},\\ \frac{2}{3}N(9a,27a,c;8(n-a)+36a+c)-2N(9a,27a,c;2(n-a)+9a+\frac{c}{4}) \\ \qquad \text{if } c\equiv 8,16 \pmod{24}. \end{cases} \end{aligned}$$

By Lemma 5.3(i),

$$N(a,27a,c;8n+28a+c)=N(9a,27a,c;8(n-a)+36a+c)$$

and for $c\equiv 0 \pmod 4$,

$$N(a,27a,c;2n+7a+c/4)=N(9a,27a,c;2(n-a)+9a+c/4).$$

Thus the result follows.

Corollary 5.1. *Let $a,c\in\mathbb{Z}^+$ with $2\nmid a$ and $c\equiv 4,20 \pmod{24}$. For $n\in\mathbb{Z}^+$ with $n\equiv \frac{1-3a}{2}+\frac{c-4}{8} \pmod 6$ we have*

$$t(a,27a,c;n)=\frac{2}{3}N(a,27a,c;8n+28a+c).$$

Proof. For $x \in \mathbb{Z}$ we have $x^2 \equiv 0, 1 \pmod 4$. Thus, $x^2 + 3y^2 \not\equiv 2 \pmod 4$. Hence, if $2m + 7a + \frac{c}{4} = ax^2 + 27ay^2 + cz^2 = a(x^2 + 3(3y)^2) + cz^2$ for some $x, y, z \in \mathbb{Z}$, then $2m + 7a + \frac{c}{4} \not\equiv 2a \pmod 4$. This yields $m \not\equiv \frac{1}{2}(a + \frac{c}{4}) \pmod 2$. Thus, for $m \equiv \frac{1}{2}(a + \frac{c}{4}) \pmod 2$ we have $N(a, 27a, c; 2m + 7a + c/4) = 0$. Since $n \equiv \frac{1-3a}{2} + \frac{c-4}{8} \pmod 6$ we see that $n \equiv \frac{a+1}{2} + \frac{c-4}{8} = \frac{1}{2}(a + \frac{c}{4}) \pmod 2$ and so $N(a, 27a, c; 2n + 7a + c/4) = 0$. Clearly $n \equiv -c \pmod 3$. Thus the result follows from Theorem 5.2.

Theorem 5.3. *Let $a, c \in \mathbb{Z}^+$ with $2 \nmid a$, $3 \nmid a$ and $6 \mid c$. For $n \in \mathbb{Z}^+$ with $n \equiv a \pmod 3$ we have*

$$t(a,27a,c;n) = \begin{cases} \frac{2}{3}N(a,27a,c;8n+28a+c) & \text{if } c \equiv 6,18 \pmod{24}, \\ \frac{2}{3}\big(N(a,27a,c;8n+28a+c) - N\big(a,27a,c;2n+7a+\frac{c}{4}\big)\big) & \text{if } c \equiv 12 \pmod{24}, \\ \frac{2}{3}N(a,27a,c;8n+28a+c) - 2N\big(a,27a,c;2n+7a+\frac{c}{4}\big) & \text{if } c \equiv 0 \pmod{24}. \end{cases}$$

Proof. By Lemma 5.1 and Theorems 4.1-4.3,

$$t(a,27a,c;n) = t\big(3a,9a,\frac{c}{3};\frac{n-a}{3}\big)$$
$$= \begin{cases} \frac{2}{3}N\big(3a,9a,\frac{c}{3};8\frac{n-a}{3}+12a+\frac{c}{3}\big) \quad \text{if } c \equiv 6,18 \pmod{24}, \\ \frac{2}{3}\big(N\big(3a,9a,\frac{c}{3};8\frac{n-a}{3}+12a+\frac{c}{3}\big) - N\big(3a,9a,\frac{c}{3};2\frac{n-a}{3}+3a+\frac{c}{12}\big)\big) \\ \qquad \text{if } c \equiv 12 \pmod{24}, \\ \frac{2}{3}N\big(3a,9a,\frac{c}{3};8\frac{n-a}{3}+12a+\frac{c}{3}\big) - 2N\big(3a,9a,\frac{c}{3};2\frac{n-a}{3}+3a+\frac{c}{12}\big) \\ \qquad \text{if } c \equiv 0 \pmod{24}. \end{cases}$$

Note that $N(3a, 9a, c/3; m) = N(9a, 27a, c; 3m) = N(a, 27a, c; 3m)$ for $m \in \mathbb{Z}^+$. We then obtain the result.

Theorem 5.4. *Suppose $n \in \mathbb{Z}^+$.*

(i) *For $n \equiv 1 \pmod 5$ we have $t(3, 9, 25; n) = \frac{1}{2}N(3, 9, 25; 8n + 37)$.*

(ii) *Suppose $a, c \in \mathbb{Z}^+$, $a \equiv 1, 3, 7, 9 \pmod{10}$, $c \equiv 2 \pmod 4$ and $n \equiv a \equiv -c \pmod 5$. Then*

$$t(3a, 25a, c; n) = \frac{2}{3}N(3a, 25a, c; 8n + 28a + c).$$

Proof. We first prove (i). Suppose $8n + 37 = 3x^2 + 9y^2 + 25z^2$ for some $x, y, z \in \mathbb{Z}$. Since $8n + 37 \equiv 8 + 37 \equiv 0 \pmod 5$ and $x^2 \equiv 0, \pm 1 \pmod 5$ we see that $x^2 + 3y^2 \equiv 0 \pmod 5$ and so $x \equiv y \equiv 0 \pmod 5$. Thus, $8n + 37 = 3(5x)^2 + 9(5y)^2 + 25z^2$ for some $x, y, z \in \mathbb{Z}$. This yields $25 \mid 8n + 37$ and $\frac{8n+37}{25} = 3x^2 + 9y^2 + z^2$. Hence, when $25 \nmid 8n + 37$ we have $N(3, 9, 25; 8n+37) = 0$ and so $t(3, 9, 25; n) = 0$ by (3.2). Now assume $25 \mid 8n+37$. From the above, (3.2) and Theorem 3.3 we deduce that for $n \equiv 1 \pmod 5$,

$$t(3,9,25;n) = \big|\{(x,y,z) \in \mathbb{Z}^3 \mid 3x^2 + 9y^2 + 25z^2 = 8n + 37,\ 2 \nmid xyz\}\big|$$
$$= \big|\{(x,y,z) \in \mathbb{Z}^3 \mid 3(5x)^2 + 9(5y)^2 + 25z^2 = 8n + 37,\ 2 \nmid xyz\}\big|$$

$$= \left|\{(x,y,z)\in\mathbb{Z}^3 \mid 3x^2+9y^2+z^2=\frac{8n+37}{25},\ 2\nmid xyz\}\right|$$
$$= t\big(1,3,9;\frac{n-36}{25}\big)=\frac{1}{2}N\big(1,3,9;\frac{8n+37}{25}\big)=\frac{1}{2}N(25,75,225;8n+37)$$
$$= \frac{1}{2}N(3\cdot 25, 9\cdot 25, 25; 8n+37)=\frac{1}{2}N(3,9,25;8n+37).$$

This proves (i).

Now we consider (ii). Suppose $a,c\in\mathbb{Z}^+$, $5\nmid a$, $c\equiv 2 \pmod 4$ and $n\equiv a\equiv -c \pmod 5$. Then $8n+28a+c\equiv 0 \pmod 5$. If $8n+28a+c=3ax^2+25ay^2+cz^2$ for some $x,y,z\in\mathbb{Z}$, we see that $a(3x^2-z^2)\equiv 3ax^2+25ay^2+cz^2=8n+28a+c\equiv 0 \pmod 5$. This yields $5\mid x$, $5\mid z$ and so $25\mid 8n+28a+c$. Hence, when $25\nmid 8n+28a+c$ we have $N(3a,25a,c;8n+28a+c)=0$ and so $t(3a,25a,c;n)=0$ by (3.2). Now assume $25\mid 8n+28a+c$. Applying (3.2) and Theorem 4.1 we see that

$$t(3a,25a,c;n)$$
$$= \left|\{(x,y,z)\in\mathbb{Z}^3 \mid 3ax^2+25ay^2+cz^2=8n+28a+c,\ 2\nmid xyz\}\right|$$
$$= \left|\{(x,y,z)\in\mathbb{Z}^3 \mid 3a(5x)^2+25ay^2+c(5z)^2=8n+28a+c,\ 2\nmid xyz\}\right|$$
$$= \left|\{(x,y,z)\in\mathbb{Z}^3 \mid 3ax^2+ay^2+cz^2=\frac{8n+28a+c}{25},\ 2\nmid xyz\}\right|$$
$$= t\big(a,3a,c;\frac{n-9a-3c}{25}\big)=\frac{2}{3}N\big(a,3a,c;\frac{8n+28a+c}{25}\big)$$
$$= \frac{2}{3}N(25a,75a,25c;8n+28a+c)=\frac{2}{3}N(25a,3a,c;8n+28a+c).$$

This completes the proof.

Using the method in the proof of Theorem 5.4(i) one can similarly prove the following theorem.

Theorem 5.5. *Let $n\in\mathbb{Z}^+$. Then*

$$t(2,25,25;n)=\frac{2}{3}N(2,25,25;8n+52)\quad \text{for } n\equiv 1 \pmod 5,$$
$$t(1,25,50;n)=\frac{2}{3}N(1,25,50;8n+76)\quad \text{for } n\equiv 3 \pmod 5,$$
$$t(3,3,49;n)=\frac{1}{2}N(3,3,49;8n+55)\quad \text{for } n\equiv 1 \pmod 7,$$
$$t(3,10,49;n)=\frac{2}{3}N(3,10,49;8n+62)\quad \text{for } n\equiv 1 \pmod 7.$$

6. Three relations between $t(a,b,c;n)$ and $N(a,b,c;8n+a+b+c)$

For $a,b,c,n\in\mathbb{Z}^+$, in this section we establish three general relations between $t(a,b,c;n)$ and $N(a,b,c;8n+a+b+c)$ under certain conditions.

Theorem 6.1. *Let $a,b,c,n\in\mathbb{Z}^+$ with $2\nmid ab$, $4\mid a-b$ and $4\mid c-2$. Then*

$$t(a,b,c;n)=N(a,b,c;8n+a+b+c)-N(a,b,c;2n+(a+b+c)/4).$$

Proof. Suppose $8n+a+b+c=ax^2+by^2+cz^2$ for some $x,y,z\in\mathbb{Z}$. Since $a\equiv b\equiv\pm1\pmod 4$ and $c\equiv 2\pmod 4$, we see that $a+b+c\equiv 0\pmod 4$ and so $a(x^2+y^2)+2z^2\equiv ax^2+by^2+cz^2=8n+a+b+c\equiv 0\pmod 4$. Therefore, either $x\equiv y\equiv z\equiv 0\pmod 2$ or $x\equiv y\equiv z\equiv 1\pmod 2$. Hence appealing to (3.2) we deduce that

$$\begin{aligned}
&t(a,b,c;n)\\
&=\left|\{(x,y,z)\in\mathbb{Z}^3\,\middle|\,8n+a+b+c=ax^2+by^2+cz^2,\ 2\nmid xyz\}\right|\\
&=N(a,b,c;8n+a+b+c)\\
&\quad-\left|\{(x,y,z)\in\mathbb{Z}^3\,\middle|\,8n+a+b+c=ax^2+by^2+cz^2,\ x\equiv y\equiv z\equiv 0\pmod 2\}\right|\\
&=N(a,b,c;8n+a+b+c)\\
&\quad-\left|\{(x,y,z)\in\mathbb{Z}^3\,\middle|\,8n+a+b+c=a(2x)^2+b(2y)^2+c(2z)^2\}\right|\\
&=N(a,b,c;8n+a+b+c)-N(a,b,c;2n+(a+b+c)/4).
\end{aligned}$$

This proves the theorem.

Theorem 6.2. *Suppose that $a,b,c,n\in\mathbb{Z}^+$ with $2\nmid ab$ and $4\mid a-b$. If $c\equiv a\pmod 4$ or $c\equiv 4\pmod 8$, then*

$$t(a,b,c;n)=N(a,b,c;8n+a+b+c).$$

Proof. Assume $c\equiv a\pmod 4$ and $8n+a+b+c=ax^2+by^2+cz^2$ for some $x,y,z\in\mathbb{Z}$. If $2\mid z$, then $3a\equiv 8n+a+b+c\equiv ax^2+by^2\equiv a(x^2+y^2)\pmod 4$ and so $x^2+y^2\equiv 3\pmod 4$. Since $x^2,y^2\equiv 0,1\pmod 4$, we must have $x^2+y^2\not\equiv 3\pmod 4$ and get a contradiction. Hence $2\nmid z$ and $a(x^2+y^2)\equiv ax^2+by^2=8n+a+b+c-cz^2\equiv a+b\equiv 2a\pmod 4$. That is, $x^2+y^2\equiv 2\pmod 4$. This implies that $2\nmid xy$.

Now assume $c\equiv 4\pmod 8$ and $8n+a+b+c=ax^2+by^2+cz^2$ for some $x,y,z\in\mathbb{Z}$. Then $a(x^2+y^2)\equiv ax^2+by^2=8n+a+b+c-cz^2\equiv a+b\equiv 2\pmod 4$. This implies $2\nmid xy$ and so $cz^2=8n+a+b+c-ax^2-by^2\equiv a+b+c-a-b=c\pmod 8$. Since $c\equiv 4\pmod 8$ we get $2\nmid z$.

By the above and (3.2), for $c\equiv a\pmod 4$ or $c\equiv 4\pmod 8$,

$$\begin{aligned}
t(a,b,c;n)&=\left|\{(x,y,z)\in\mathbb{Z}^3\,\middle|\,8n+a+b+c=ax^2+by^2+cz^2,\ 2\nmid xyz\}\right|\\
&=N(a,b,c;8n+a+b+c).
\end{aligned}$$

This proves the theorem.

Theorem 6.3. *Let $a,b,c,n\in\mathbb{Z}^+$ with $2\nmid a$, $2\mid b$, $2\mid c$, $8\nmid b$, $8\nmid c$ and $8\nmid b+c$. Then $t(a,b,c;n)=N(a,b,c;8n+a+b+c)$.*

Proof. Suppose $8n+a+b+c=ax^2+by^2+cz^2$ for some $x,y,z\in\mathbb{Z}$. Then clearly $2\nmid x$ and so $by^2+cz^2=8n+a+b+c-ax^2\equiv b+c\pmod 8$. If $2\mid y$, since $8\nmid b+c$ we have $2\nmid z$ and so $c\equiv cz^2\equiv by^2+cz^2\equiv b+c\pmod 8$. This contradicts the assumption $8\nmid b$. Hence $2\nmid y$. Similarly, $2\nmid z$. Now applying (3.2) yields the result.

Theorem 6.4. *Suppose $n\in\mathbb{Z}^+$. Then n is represented by $\frac{x(x+1)}{2}+\frac{y(y+1)}{2}+9\frac{z(z+1)}{2}$ if and only if $n\not\equiv 5,8\pmod 9$.*

Proof. By Theorem 6.2, $t(1,1,9;n)=N(1,1,9;8n+11)$. By Three Squares Theorem, $8n+11=x^2+y^2+z^2$ for some $x,y,z\in\mathbb{Z}$. Assume $n\equiv 0,1\pmod 3$. Then $3\nmid 8n+11$.

Since $m^2 \equiv 0, 1 \pmod 3$ for $m \in \mathbb{Z}$ we must have $3 \mid xyz$ and so $8n + 11$ is represented by $x^2 + y^2 + 9z^2$. Hence $t(1, 1, 9; n) = N(1, 1, 9; 8n + 11) \geq 1$. For $n \equiv 2 \pmod 9$ we have $9 \mid 8n + 11$ and $\frac{8n+11}{9} \equiv 3 \pmod 8$. Hence $\frac{8n+11}{9} = x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{Z}$. This yields $8n + 11 = (3x)^2 + (3y)^2 + 9z^2$. Therefore $t(1, 1, 9; n) = N(1, 1, 9; 8n + 11) \geq 1$. Finally we assume $n \equiv 5, 8 \pmod 9$. If $8n + 11 = x^2 + y^2 + 9z^2$ for some integers x, y and z, since $m^2 \equiv 0, 1, 4, 7 \pmod 9$ for $m \in \mathbb{Z}$ we see that $x^2 + y^2 \equiv 0, 1, 2, 4, 5, 7, 8 \pmod 9$ and so $8n + 11 = x^2 + y^2 + 9z^2 \not\equiv 3, 6 \pmod 9$. This yields $n \not\equiv 5, 8 \pmod 9$, which contradicts the assumption. Hence $t(1, 1, 9; n) = N(1, 1, 9; 8n + 11) = 0$ for $n \equiv 5, 8 \pmod 9$. Putting the above together proves the theorem.

Let $a, b, c, n \in \mathbb{Z}^+$. By (3.2), $t(a, b, c; n) \leq N(a, b, c; 8n + a + b + c)$. As stated in Theorems 6.2 and 6.3, under some certain conditions we have $t(a, b, c; n) = N(a, b, c; 8n + a+b+c)$. We have more interest in the non-trivial case $t(a, b, c; n) < N(a, b, c; 8n+a+b+c)$. Assume $t(a, b, c; n) < N(a, b, c; 8n+a+b+c)$. Based on calculations with $a \leq 7$, $a \leq b \leq 30$ and $b \leq c \leq 50$ on Maple, we pose the following challenging conjectures.

Conjecture 6.1. *Let $n \in \mathbb{Z}^+$. For $(a, b, c) = (1, 1, 3), (1, 1, 4), (1, 1, 6), (1, 1, 7), (1, 1, 15), (1, 2, 2), (1, 2, 5), (1, 3, 3), (1, 3, 9), (1, 5, 10), (1, 6, 9), (1, 7, 7), (1, 7, 15), (1, 9, 15), (1, 15, 15), (1, 15, 25), (2, 3, 3)$ we have*

$$t(a, b, c; n) = \frac{1}{2}\Big(N(a, b, c; 4(8n + a + b + c)) - N(a, b, c; 8n + a + b + c)\Big).$$

Conjecture 6.2. *Let $n \in \mathbb{Z}^+$.*
(i) *For even n and $(a, b, c) = (1, 2, 15), (1, 15, 18), (1, 15, 30), (3, 10, 45)$ we have*

$$t(a, b, c; n) = \frac{1}{2}\Big(N(a, b, c; 4(8n + a + b + c)) - N(a, b, c; 8n + a + b + c)\Big).$$

(ii) *For odd n and $(a, b, c) = (1, 6, 7), (1, 7, 42), (2, 3, 21), (2, 9, 15), (3, 5, 6), (3, 5, 10), (5, 21, 35)$ we have*

$$t(a, b, c; n) = \frac{1}{2}\Big(N(a, b, c; 4(8n + a + b + c)) - N(a, b, c; 8n + a + b + c)\Big).$$

Conjecture 6.3. *Let $n \in \mathbb{Z}^+$. For $(a, b, c) = (1, 3, 5), (1, 3, 7), (1, 3, 15), (1, 3, 21), (1, 5, 15), (1, 7, 21), (3, 5, 9), (3, 5, 15), (3, 7, 21)$ we have*

$$t(a, b, c; n) = \frac{1}{2}\Big(3N(a, b, c; 8n + a + b + c) - N(a, b, c; 4(8n + a + b + c))\Big).$$

Conjecture 6.4. *Let $n \in \mathbb{Z}^+$.*
(i) *For even n and $(a, b, c) = (1, 6, 15), (1, 10, 15)$ we have*

$$t(a, b, c; n) = \frac{1}{2}\Big(3N(a, b, c; 8n + a + b + c) - N(a, b, c; 4(8n + a + b + c))\Big).$$

(ii) *For odd n and $(a, b, c) = (1, 2, 7), (1, 7, 14), (2, 3, 5), (3, 5, 30)$ we have*

$$t(a, b, c; n) = \frac{1}{2}\Big(3N(a, b, c; 8n + a + b + c) - N(a, b, c; 4(8n + a + b + c))\Big).$$

Conjectures 6.1-6.4 have been checked for $n \leq 150$ with Maple.

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