

LINEAR RECURSIVE SEQUENCES AND POWERS OF MATRICES

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1. INTRODUCTION

In this paper we study the properties of linear recursive sequences and give some applications to matrices.

For $\alpha_1, \alpha_2 \in \mathbb{Z}$, the corresponding Lucas sequence $\{u_n\}$ is given by $u_0 = 0$, $u_1 = 1$, and $u_{n+1} + \alpha_1 u_n + \alpha_2 u_{n-1} = 0$ ($n \geq 1$). Such series have very interesting properties and applications, and have been studied in great detail by Lucas and later writers (cf. [2], [4], [6], [10]).

The general linear recursive sequences $\{u_n\}$ is defined by $u_n + \alpha_1 u_{n-1} + \dots + \alpha_m u_{n-m} = 0$ ($n \geq 0$). Since Dickson [2], many mathematicians have been devoted to the study of the theory of linear recursive sequences. More recently, linear recursive sequences in finite fields have often been considered; for references, one may consult [3], [5], [7], [8], [11], [12], [13], [16], [17], and [18].

In this paper we extend the Lucas series to general linear recursive sequences by defining $\{u_n(\alpha_1, \dots, \alpha_m)\}$ as follows:

$$\begin{aligned} u_{1-m} = \dots = u_{-1} = 0, \quad u_0 = 1, \\ u_n + \alpha_1 u_{n-1} + \dots + \alpha_m u_{n-m} = 0 \quad (n = 0, \pm 1, \pm 2, \dots), \end{aligned} \quad (1.1)$$

where $m \geq 2$ and $\alpha_m \neq 0$.

We mention that sequences like (1.1) have been studied by Somer in [12] and [13], and by Wagner in [15].

In Section 2 we obtain various expressions for $\{u_n(\alpha_1, \dots, \alpha_m)\}$. For example,

$$\begin{aligned} u_n(\alpha_1, \dots, \alpha_m) &= \sum_{k_1+2k_2+\dots+mk_m=n} \frac{(k_1+\dots+k_m)!}{k_1! \dots k_m!} (-1)^{k_1+\dots+k_m} \alpha_1^{k_1} \dots \alpha_m^{k_m} \\ &= \sum_{i=1}^m \frac{\lambda_i^{n+m-1}}{\prod_{j \neq i} (\lambda_i - \lambda_j)} \quad (n = 0, 1, 2, \dots), \end{aligned}$$

where $\lambda_1, \dots, \lambda_m$ are all distinct roots of the equation $x^m + \alpha_1 x^{m-1} + \dots + \alpha_m = 0$.

The purpose of Section 3 is to give the formula for the powers of a square matrix and further properties of $\{u_n(\alpha_1, \dots, \alpha_m)\}$. The main result is that

$$A^n = \sum_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} \alpha_{s-r} u_{n-s} \right) A^r, \quad (1.2)$$

where $u_n = u_n(\alpha_1, \dots, \alpha_m)$ ($n = 0, \pm 1, \pm 2, \dots$) and A is an $m \times m$ matrix with the characteristic polynomial $\alpha_0 x^m + \alpha_1 x^{m-1} + \dots + \alpha_m$ ($\alpha_0 = 1$).

Formula (1.2) is a generalization of the Hamilton-Cayley theorem, and it provides a simple method of calculating the powers of a square matrix.

Let $\lambda_1, \dots, \lambda_m$ be the roots of the equation $x^m + \alpha_1 x^{m-1} + \dots + \alpha_m = 0$, $u_n = u_n(\alpha_1, \dots, \alpha_m)$, and $s_n = \lambda_1^n + \dots + \lambda_m^n$ ($n = 1, 2, 3, \dots$). In Sections 2 and 3 we also show that

$$\sum_{k=1}^n s_k u_{n-k} = nu_n \quad \text{and} \quad s_n = -\sum_{k=1}^m ka_k u_{n-k}. \tag{1.3}$$

We establish the following identity in Section 4:

$$u_{kn+l} = \sum_{k_0+k_1+\dots+k_{m-1}=k} \frac{k!}{k_0!k_1!\dots k_{m-1}!} \prod_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right)^{k_r} u_{n-1}^{\sum_{r=0}^{m-1} r k_r + l}, \tag{1.4}$$

where $u_r = u_r(a_1, \dots, a_m)$ and $a_0 = 1$.

For later convenience, we use the following notations throughout this paper: \mathbb{Z} denotes the set of integers; \mathbb{Z}^+ denotes the set of positive integers; $|A|$ denotes the determinant of A ; and $\{u_n(a_1, \dots, a_m)\}$ denotes the sequence defined by (1.1).

2. EXPRESSIONS FOR $\{u_n(a_1, \dots, a_m)\}$

In this section we establish some formulas for $\{u_n(a_1, \dots, a_m)\}$.

Lemma 2.1: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0$. For any $n \in \mathbb{Z}$, we have

$$u_n(a_1, \dots, a_m) = -\frac{1}{a_m} u_{-n-m} \left(\frac{a_{m-1}}{a_m}, \dots, \frac{a_1}{a_m}, \frac{1}{a_m} \right).$$

Proof: Let

$$v_n = u_n \left(\frac{a_{m-1}}{a_m}, \dots, \frac{a_1}{a_m}, \frac{1}{a_m} \right) \quad \text{and} \quad u_n = -\frac{1}{a_m} v_{-n-m}.$$

Since $v_{1-m} = \dots = v_{-1} = 0$, $v_{-m} = -a_m v_0 = -a_m$, we see that $u_{1-m} = \dots = u_{-1} = 0$, $u_0 = 1$. Also,

$$\begin{aligned} & u_n + a_1 u_{n-1} + \dots + a_m u_{n-m} \\ &= -\left(\frac{1}{a_m} v_{-n-m} + \frac{a_1}{a_m} v_{-n-m+1} + \dots + \frac{a_{m-1}}{a_m} v_{-n-1} + v_{-n} \right) \\ &= 0 \quad (n = 0, \pm 1, \pm 2, \dots). \end{aligned}$$

Thus, $u_n = u_n(a_1, \dots, a_m)$ for any $n \in \mathbb{Z}$.

Theorem 2.1: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0$. Then the generating functions of $\{u_n(a_1, \dots, a_m)\}$ and $\{u_{-n}(a_1, \dots, a_m)\}$ are given by

$$\sum_{n=0}^{\infty} u_n(a_1, \dots, a_m) x^n = \frac{1}{1 + a_1 x + \dots + a_m x^m}$$

and

$$\sum_{n=0}^{\infty} u_{-n}(a_1, \dots, a_m) x^n = 1 - \frac{x^m}{x^m + a_1 x^{m-1} + \dots + a_m}.$$

Proof: Let $u_n = u_n(a_1, \dots, a_m)$, $a_0 = 1$, and $a_k = 0$ for $k > m$. Then

$$\left(\sum_{n=0}^{\infty} u_n x^n \right) \left(\sum_{k=0}^m a_k x^k \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k u_{n-k} \right) x^n.$$

Observe that $a_{m+1} = \dots = a_n = 0$ for $n > m$ and that $u_{n-m} = \dots = u_{-1} = 0$ for $n \in \{1, 2, \dots, m-1\}$. So we have

$$\sum_{k=0}^n a_k u_{n-k} = \sum_{k=0}^m a_k u_{n-k} = 0 \quad \text{for } n = 1, 2, 3, \dots,$$

and therefore,

$$\left(\sum_{n=0}^{\infty} u_n x^n \right) \left(\sum_{k=0}^m a_k x^k \right) = a_0 u_0 = 1.$$

It then follows that

$$\sum_{n=0}^{\infty} u_n x^n = \frac{1}{1 + a_1 x + \dots + a_m x^m}.$$

From the above and Lemma 2.1, we see that

$$\begin{aligned} \sum_{n=1}^{\infty} u_{-n} x^n &= -\frac{1}{a_m} \sum_{n=m}^{\infty} u_{n-m} \left(\frac{a_{m-1}}{a_m}, \dots, \frac{a_1}{a_m}, \frac{1}{a_m} \right) x^n \\ &= -\frac{1}{a_m} x^m \sum_{k=0}^{\infty} u_k \left(\frac{a_{m-1}}{a_m}, \dots, \frac{a_1}{a_m}, \frac{1}{a_m} \right) x^k \\ &= -\frac{x^m}{a_m} \cdot \frac{1}{1 + \frac{a_{m-1}}{a_m} x + \dots + \frac{1}{a_m} x^m} \\ &= -\frac{x^m}{x^m + a_1 x^{m-1} + \dots + a_m}. \end{aligned}$$

This completes the proof.

Corollary 2.1: Let $a_0 = b_0 = 1$ and $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n) = 1$. For $m = 1, 2, 3, \dots$, we have $b_m = u_m(a_1, \dots, a_m)$.

Proof: Since the coefficient of x^m in $(1 + a_1 x + \dots + a_m x^m + \dots)^{-1}$ is the same as the coefficient of x^m in $(1 + a_1 x + \dots + a_m x^m)^{-1}$, by using Theorem 2.1 we get $b_m = u_m(a_1, \dots, a_m)$. This completes the proof.

We remark that Corollary 2.1 gives a simple method of calculating $\{b_n\}$.

Theorem 2.2: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0$ and

$$x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m = \prod_{i=1}^m (x - \lambda_i).$$

(a) For $n = 0, 1, 2, \dots$, we have

$$\begin{aligned} u_n(a_1, \dots, a_m) &= \sum_{k_1+k_2+\dots+k_m=n} \lambda_1^{k_1} \lambda_2^{k_2} \dots \lambda_m^{k_m} \\ &= \sum_{k_1+2k_2+\dots+mk_m=n} \frac{(k_1 + \dots + k_m)!}{k_1! \dots k_m!} (-1)^{k_1+\dots+k_m} a_1^{k_1} \dots a_m^{k_m}. \end{aligned}$$

(b) For $n = m, m + 1, m + 2, \dots$, we have

$$\begin{aligned} u_{-n}(a_1, \dots, a_m) &= -\frac{1}{a_m} \sum_{k_1 + \dots + k_m = n-m} \frac{1}{\lambda_1^{k_1} \dots \lambda_m^{k_m}} \\ &= \sum_{k_1 + 2k_2 + \dots + mk_m = n-m} \frac{(k_1 + \dots + k_m)!}{k_1! \dots k_m!} \left(-\frac{1}{a_m}\right)^{k_1 + \dots + k_m + 1} a_1^{k_{m-1}} \dots a_m^{k_1}. \end{aligned}$$

Proof: Since $1 + a_1x + \dots + a_mx^m = (1 - \lambda_1x) \dots (1 - \lambda_mx)$, by Theorem 2.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(a_1, \dots, a_m)x^n &= \prod_{i=1}^m \frac{1}{1 - \lambda_i x} = \prod_{i=1}^m \left(\sum_{k=0}^{\infty} \lambda_i^k x^k \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k_1 + \dots + k_m = n} \lambda_1^{k_1} \dots \lambda_m^{k_m} \right) x^n. \end{aligned}$$

This implies

$$u_n(a_1, \dots, a_m) = \sum_{k_1 + k_2 + \dots + k_m = n} \lambda_1^{k_1} \lambda_2^{k_2} \dots \lambda_m^{k_m}.$$

From Theorem 2.1 and the multinomial theorem, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(a_1, \dots, a_m)x^n &= \frac{1}{1 + a_1x + \dots + a_mx^m} = \sum_{r=0}^{\infty} (-1)^r (a_1x + \dots + a_mx^m)^r \\ &= \sum_{r=0}^{\infty} (-1)^r \sum_{n=0}^{\infty} \left(\sum_{\substack{k_1 + 2k_2 + \dots + mk_m = n \\ k_1 + \dots + k_m = r}} \frac{r!}{k_1! \dots k_m!} a_1^{k_1} \dots a_m^{k_m} \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k_1 + 2k_2 + \dots + mk_m = n} \frac{(k_1 + \dots + k_m)!}{k_1! \dots k_m!} (-1)^{k_1 + \dots + k_m} a_1^{k_1} \dots a_m^{k_m} \right) x^n. \end{aligned}$$

Thus,

$$u_n(a_1, \dots, a_m) = \sum_{k_1 + 2k_2 + \dots + mk_m = n} \frac{(k_1 + \dots + k_m)!}{k_1! \dots k_m!} (-1)^{k_1 + \dots + k_m} a_1^{k_1} \dots a_m^{k_m}.$$

This proves part (a).

Now consider part (b). It follows from Theorem 2.1 that

$$\begin{aligned} \sum_{n=m}^{\infty} u_{-n}(a_1, \dots, a_m)x^n &= -x^m \frac{1}{(x - \lambda_1)} \dots \frac{1}{(x - \lambda_m)} \\ &= \frac{(-1)^{m-1} x^m}{\lambda_1 \dots \lambda_m} \cdot \frac{1}{(1 - \frac{x}{\lambda_1})} \dots \frac{1}{(1 - \frac{x}{\lambda_m})} = -\frac{x^m}{a_m} \prod_{i=1}^m \left(\sum_{k=0}^{\infty} \left(\frac{x}{\lambda_i} \right)^k \right) \\ &= -\frac{1}{a_m} \sum_{n=m}^{\infty} \left(\sum_{k_1 + \dots + k_m = n-m} \left(\frac{1}{\lambda_1} \right)^{k_1} \dots \left(\frac{1}{\lambda_m} \right)^{k_m} \right) x^n. \end{aligned}$$

Therefore, we have

$$u_{-n}(a_1, \dots, a_m) = -\frac{1}{a_m} \sum_{k_1 + \dots + k_m = n-m} \frac{1}{\lambda_1^{k_1} \dots \lambda_m^{k_m}} \text{ for } n \geq m.$$

By Lemma 2.1 and part (a),

$$\begin{aligned} u_{-n}(a_1, \dots, a_m) &= -\frac{1}{a_m} u_{n-m} \left(\frac{a_{m-1}}{a_m}, \dots, \frac{a_1}{a_m}, \frac{1}{a_m} \right) \\ &= -\frac{1}{a_m} \sum_{k_1 + 2k_2 + \dots + mk_m = n-m} \frac{(k_1 + \dots + k_m)!}{k_1! \dots k_m!} (-1)^{k_1 + \dots + k_m} \left(\frac{a_{m-1}}{a_m} \right)^{k_1} \dots \left(\frac{1}{a_m} \right)^{k_m}. \end{aligned}$$

Hence, the proof is complete.

Remark 2.1: Let $x^m + a_1x^{m-1} + \dots + a_m = (x - \lambda_1) \dots (x - \lambda_m)$. If $\{u_n(a_1, \dots, a_m)\}$ is given by its generating function, by Theorem 2.2(a) we have

$$u_n(a_1, \dots, a_m) = \sum_{k_1 + k_2 + \dots + k_m = n} \lambda_1^{k_1} \lambda_2^{k_2} \dots \lambda_m^{k_m} \quad (n \geq 0), \tag{2.1}$$

as was found by Wagner [15].

Suppose $a_0 = 1$ and $a_k = 0$ for $k \notin \{0, 1, \dots, m\}$. Using Theorem 2.1 and Cramer's rule, one can prove the following facts:

(a) For $n = 1, 2, 3, \dots$, we have

$$u_n(a_1, \dots, a_m) = (-1)^n \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_0 & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2-n} & a_{3-n} & \dots & a_1 \end{vmatrix}. \tag{2.2}$$

(b) For $n = m+1, m+2, \dots$, we have

$$u_{-n}(a_1, \dots, a_m) = \left(-\frac{1}{a_m} \right)^{n-m+1} \begin{vmatrix} a_{m-1} & a_{m-2} & \dots & a_{2m-n} \\ a_m & a_{m-1} & \dots & a_{2m-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2} & a_{n-3} & \dots & a_{m-1} \end{vmatrix}. \tag{2.3}$$

Here, (a) is well known (see [9]) when $\{u_n(a_1, \dots, a_m)\}$ is given by its generating function.

Theorem 2.3: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0$, and $\lambda_1, \lambda_2, \dots, \lambda_m$ be the distinct roots of the equation $x^m + a_1x^{m-1} + \dots + a_m = 0$. For any integer n , we have

$$u_n(a_1, \dots, a_m) = \sum_{i=1}^m \frac{\lambda_i^{n+m-1}}{\prod_{\substack{j=1 \\ j \neq i}}^m (\lambda_i - \lambda_j)}.$$

Proof: Consider the following system of m linear equations in m unknowns x_1, x_2, \dots, x_m :

$$\begin{aligned} x_1 + x_2 + \dots + x_m &= 0 \\ \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_m x_m &= 0 \\ \dots & \\ \lambda_1^{m-2} x_1 + \lambda_2^{m-2} x_2 + \dots + \lambda_m^{m-2} x_m &= 0 \\ \lambda_1^{m-1} x_1 + \lambda_2^{m-1} x_2 + \dots + \lambda_m^{m-1} x_m &= 1. \end{aligned} \tag{2.4}$$

Since (2.4) is equivalent to

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-2} & \lambda_2^{m-2} & \dots & \lambda_m^{m-2} \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \dots & \lambda_m^{m-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

by the solution of Vandermonde's determinants and Cramer's rule, we obtain

$$\begin{aligned} x_i &= \frac{1}{\prod_{r>s} (\lambda_r - \lambda_s)} \begin{vmatrix} 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_{i-1} & 0 & \lambda_{i+1} & \dots & \lambda_m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \dots & \lambda_{i-1}^{m-1} & 1 & \lambda_{i+1}^{m-1} & \dots & \lambda_m^{m-1} \end{vmatrix} \\ &= \frac{(-1)^{m+i}}{\prod_{r>s} (\lambda_r - \lambda_s)} \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_{i-1} & \lambda_{i+1} & \dots & \lambda_m \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-2} & \dots & \lambda_{i-1}^{m-2} & \lambda_{i+1}^{m-2} & \dots & \lambda_m^{m-2} \end{vmatrix} \\ &= \frac{(-1)^{m+i}}{\prod_{r>s} (\lambda_r - \lambda_s)} \prod_{\substack{r>s \\ r, s \neq i}} (\lambda_r - \lambda_s) = \frac{1}{\prod_{j \neq i} (\lambda_i - \lambda_j)} \quad (i = 1, 2, \dots, m). \end{aligned}$$

For $n \in \mathbb{Z}$, set

$$u_n = \sum_{i=1}^m \frac{\lambda_i^{n+m-1}}{\prod_{j \neq i} (\lambda_i - \lambda_j)}.$$

From the above, we see that $u_{1-m} = \dots = u_{-1} = 0$, $u_0 = 1$. Also,

$$\begin{aligned} u_n + a_1 u_{n-1} + \dots + a_m u_{n-m} &= \sum_{i=1}^m \frac{\lambda_i^{n-1}}{\prod_{j \neq i} (\lambda_i - \lambda_j)} (\lambda_i^n + a_1 \lambda_i^{m-1} + \dots + a_m) \\ &= 0 \quad (n = 0, \pm 1, \pm 2, \dots). \end{aligned}$$

Thus, $u_n = u_n(a_1, \dots, a_m)$ for $n = 0, \pm 1, \pm 2, \dots$. This completes the proof.

For example, let $\{S(n, m)\}$ be the Stirling numbers of the second kind given by

$$x^n = \sum_{m=0}^n S(n, m) x(x-1) \dots (x-m+1).$$

It is well known (see [1]) that

$$S(n, m) = \frac{1}{m!} \sum_{i=0}^m \binom{m}{i} (-1)^{m-i} i^n = \sum_{i=1}^m \frac{i^{n-1}}{\prod_{\substack{j=1 \\ j \neq i}} (i-j)} \quad \text{for } n \geq m \geq 1.$$

Thus, for $n \geq m \geq 1$, $S(n, m) = u_{n-m}(a_1, \dots, a_m)$, where a_1, \dots, a_m are determined by $(x-1)(x-2) \dots (x-m) = x^m + a_1 x^{m-1} + \dots + a_m$. From this, we may extend the Stirling numbers of the second kind by defining $S(n, m) = u_{n-m}(a_1, \dots, a_m)$ for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$.

Remark 2.2: Suppose that the equation $x^m + a_1x^{m-1} + \dots + a_m = 0$ has distinct nonzero roots $\lambda_1, \dots, \lambda_m$, and that $\{U_n\}$ satisfies the recurrence relation $U_n + a_1U_{n-1} + \dots + a_mU_{n-m} = 0$ ($n \geq m$). It is well known (see [1]) that there are m constants c_1, \dots, c_m such that $U_n = c_1\lambda_1^n + c_2\lambda_2^n + \dots + c_m\lambda_m^n$ for every $n = 0, 1, 2, \dots$.

If $a_m \neq 0$ and $x^m + a_1x^{m-1} + \dots + a_m = (x - \lambda_1)^{n_1} \dots (x - \lambda_r)^{n_r}$, where $\lambda_1, \dots, \lambda_r$ are all distinct, then using Theorem 2.1 we can prove that

$$u_n(a_1, \dots, a_m) = \frac{1}{a_m} \sum_{i=1}^r \sum_{j=0}^{n_i-1} \binom{n_i - j - 1 + n}{n} (-1)^{n_i-j} \frac{f_i^{(j)}(\frac{1}{\lambda_i})}{j!} \lambda_i^{n+n_i-j} \quad (n \geq 0), \tag{2.5}$$

where

$$f_i(x) = \prod_{\substack{s=1 \\ s \neq i}}^r \left(x - \frac{1}{\lambda_s}\right)^{-n_s} \quad \text{and} \quad f_i^{(j)}(x) = \frac{d^j f_i(x)}{dx^j}.$$

Theorem 2.4: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0$, $x^m + a_1x^{m-1} + \dots + a_m = (x - \lambda_1) \dots (x - \lambda_m)$, $s_n = \lambda_1^n + \lambda_2^n + \dots + \lambda_m^n$ and $u_n = u_n(a_1, \dots, a_m)$. For $n = 1, 2, 3, \dots$, we have

$$\sum_{k=1}^n s_k u_{n-k} = nu_n \quad \text{and} \quad \sum_{k=1}^n s_{-k} u_{k-n-m} = nu_{-n-m}.$$

Proof: Since

$$\sum_{n=0}^{\infty} u_n x^n = \frac{1}{1 + a_1x + \dots + a_mx^m} = (1 - \lambda_1x)^{-1} (1 - \lambda_2x)^{-1} \dots (1 - \lambda_mx)^{-1},$$

we have

$$\log \sum_{n=0}^{\infty} u_n x^n = -\sum_{i=1}^m \log(1 - \lambda_i x) = \sum_{i=1}^m \sum_{n=1}^{\infty} \frac{\lambda_i^n x^n}{n} = \sum_{n=1}^{\infty} \frac{s_n x^n}{n}.$$

By differentiating the expansion, we get

$$\frac{\sum_{n=1}^{\infty} nu_n x^{n-1}}{\sum_{n=0}^{\infty} u_n x^n} = \sum_{n=1}^{\infty} s_n x^{n-1}.$$

That is,

$$\left(\sum_{n=1}^{\infty} s_n x^n\right) \left(\sum_{n=0}^{\infty} u_n x^n\right) = \sum_{n=1}^{\infty} nu_n x^n.$$

Comparing the coefficients of x^n on both sides gives

$$\sum_{k=1}^n s_k u_{n-k} = nu_n.$$

To complete the proof, by the above and Lemma 2.1 one can easily derive

$$\sum_{k=1}^n s_{-k} u_{k-n-m} = nu_{-n-m}.$$

3. THE FORMULA FOR THE POWERS OF A SQUARE MATRIX

This section is devoted to giving a formula for the powers of a square matrix. First, we derive an explicit formula for companion matrices and then give a formula for arbitrary square matrices.

Theorem 3.1: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0$, $n \in \mathbb{Z}$, and $u_n = u_n(a_1, \dots, a_m)$. Then

$$\begin{pmatrix} 0 & & & -a_m \\ 1 & 0 & & -a_{m-1} \\ & 1 & & \vdots \\ & & \ddots & 0 \\ & & & 1 & -a_1 \end{pmatrix}^n = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{m-1} \\ & 1 & a_1 & \cdots & a_{m-2} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & a_1 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} u_n & u_{n+1} & \cdots & u_{n+m-1} \\ u_{n-1} & u_n & \cdots & u_{n+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-m+1} & u_{n-m+2} & \cdots & u_n \end{pmatrix}.$$

Proof: Let

$$A = \begin{pmatrix} 0 & & & -a_m \\ 1 & 0 & & -a_{m-1} \\ & 1 & & \vdots \\ & & \ddots & 0 \\ & & & 1 & -a_1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{m-1} \\ & 1 & a_1 & \cdots & a_{m-2} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & a_1 \\ & & & & 1 \end{pmatrix},$$

and

$$M_n = \begin{pmatrix} u_n & u_{n+1} & \cdots & u_{n+m-1} \\ u_{n-1} & u_n & \cdots & u_{n+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-m+1} & u_{n-m+2} & \cdots & u_n \end{pmatrix}.$$

Since $u_{1-m} = \dots = u_{-1} = 0$ and $u_0 = 1$, we see that $DM_0 = A^0$.

Clearly, $M_k A = M_{k+1}$ for any $k \in \mathbb{Z}$. Therefore, for $n = 1, 2, 3, \dots$, we have

$$M_n = M_{n-1} A = M_{n-2} A^2 = \dots = M_0 A^n$$

and

$$M_{-n} = M_{-n+1} A^{-1} = M_{-n+2} A^{-2} = \dots = M_0 A^{-n}.$$

From this, it follows that

$$DM_n = DM_0 A^n = A^n \quad \text{and} \quad DM_{-n} = DM_0 A^{-n} = A^{-n},$$

which proves the theorem.

Remark 3.1: Let $\{u_n(a_1, \dots, a_m)\}$ be given by its generating function. For $n \geq 0$, the result of Theorem 3.1 is known (see [9]).

Corollary 3.1: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0$, $n \in \mathbb{Z}$, and $u_n = u_n(a_1, \dots, a_m)$. Then

$$\begin{vmatrix} u_n & u_{n+1} & \cdots & u_{n+m-1} \\ u_{n-1} & u_n & \cdots & u_{n+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-m+1} & u_{n-m+2} & \cdots & u_n \end{vmatrix} = (-1)^{mn} a_m^n.$$

Proof: Let $A, D,$ and M_n be the matrices as in the proof of Theorem 3.1. It is clear that $|A| = (-1)^m a_m$ and $|D| = 1$. Thus, taking the determinant of both sides of the identity $A^n = DM_n$ gives the result.

Clearly, Corollary 3.1 is a vast generalization of the known fact that $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$, where $\{F_n\}$ is the Fibonacci sequence.

Corollary 3.2: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0, x^m + a_1x^{m-1} + \dots + a_m = (x - \lambda_1) \dots (x - \lambda_m), n \in \mathbb{Z}, u_n = u_n(a_1, \dots, a_m),$ and $s_n = \lambda_1^n + \lambda_2^n + \dots + \lambda_m^n$. Then

$$s_n = -\sum_{k=1}^m ka_k u_{n-k}.$$

Proof: Suppose that A is the companion matrix in Theorem 3.1. Then $x^m + a_1x^{m-1} + \dots + a_m$ is the characteristic polynomial of A and hence $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A . From matrix theory, we know that the eigenvalues of A^n are $\lambda_1^n, \lambda_2^n, \dots, \lambda_m^n$. Denote the trace of the matrix C by $\text{tr}(C)$. Then, by the above and Theorem 3.1,

$$\begin{aligned} s_n &= \lambda_1^n + \lambda_2^n + \dots + \lambda_m^n = \text{tr}(A^n) = \text{tr}(DM_n) \\ &= \sum_{i=1}^m \left(\sum_{k=0}^{m-i} a_k u_{n-k} \right) = \sum_{k=0}^{m-1} (m-k) a_k u_{n-k} \\ &= -ma_m u_{n-m} - \sum_{k=0}^{m-1} ka_k u_{n-k} = -\sum_{k=1}^m ka_k u_{n-k}. \end{aligned}$$

This proves the corollary.

Theorem 3.2: Let A be an $m \times m$ matrix with the characteristic polynomial $\chi_A(x) = a_0x^m + a_1x^{m-1} + \dots + a_m, a_m \neq 0, n \in \mathbb{Z},$ and $u_n = u_n(a_1, \dots, a_m)$. Then

$$A^n = \sum_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right) A^r.$$

Proof: For $n \in \mathbb{Z}$ and arbitrary numbers v_0, \dots, v_{m-1} , set

$$v_n^* = \sum_{s=0}^{m-1} \left(\sum_{r=0}^s a_{s-r} v_r \right) u_{n-s}.$$

Then

$$\sum_{k=0}^m a_k v_{n-k}^* = \sum_{s=0}^{m-1} \left(\sum_{r=0}^s a_{s-r} v_r \right) \sum_{k=0}^m a_k u_{n-s-k} = 0 \quad (n = 0, \pm 1, \pm 2, \dots). \tag{3.1}$$

Since $a_0 = 1$ and $u_{-1} = \dots = u_{1-m} = 0$, we see that

$$\begin{aligned} v_n^* &= \sum_{s=0}^n \left(\sum_{r=0}^s a_{s-r} v_r \right) u_{n-s} = \sum_{r=0}^n \left(\sum_{s=r}^n a_{s-r} u_{n-s} \right) v_r = v_n + \sum_{r=0}^{n-1} \left(\sum_{s=r}^n a_{s-r} u_{n-s} \right) v_r \\ &= v_n + \sum_{r=0}^{n-1} \left(\sum_{s=r}^{m+r} a_{s-r} u_{n-s} \right) v_r = v_n \quad (n = 0, 1, \dots, m-1). \end{aligned} \tag{3.2}$$

Hence, $\{v_n^*\}$ is uniquely determined by (3.1) and (3.2).

From the Hamilton-Cayley theorem, we know that $A^m + a_1A^{m-1} + \dots + a_mI = O$, where I is the $m \times m$ unit matrix and O is the $m \times m$ zero matrix. So, for $n \in \mathbb{Z}$, $A^n + a_1A^{n-1} + \dots + a_mA^{n-m} = O$. If we set $A^n = (a_{ij}^{(n)})_{m \times m}$, then

$$a_{ij}^{(n)} + a_1a_{ij}^{(n-1)} + \dots + a_ma_{ij}^{(n-m)} = 0 \quad (n = 0, \pm 1, \pm 2, \dots).$$

Applying the above result, we get

$$a_{ij}^{(n)} = \sum_{s=0}^{m-1} \left(\sum_{r=0}^s a_{s-r} a_{ij}^{(r)} \right) u_{n-s} = \sum_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right) a_{ij}^{(r)} \quad (i, j = 1, 2, \dots, m).$$

Hence,

$$A^n = \sum_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right) A^r.$$

The proof is now complete.

Since $u_{1-m} = \dots = u_{-1} = 0$ and $u_0 = 1$, we see that $a_0u_{m-r} + \dots + a_{m-1-r}u_1 = -a_{m-r}$ if $0 \leq r \leq m-1$. Thus, the Hamilton-Cayley theorem is a special result of Theorem 3.2 in the case $n = m$.

We remark that the result of Theorem 3.2 provides a very simple method of calculating the powers of a square matrix.

Corollary 3.3: Let p be an odd prime, $a, b, c, d \in \mathbb{Z}$, $p \nmid ad - bc$, $\Delta = (a - d)^2 + 4bc$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p - \left(\frac{\Delta}{p}\right)} \equiv \begin{cases} I & (\text{mod } p) \quad \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \frac{a+d}{2} I & (\text{mod } p) \quad \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad - bc)I & (\text{mod } p) \quad \text{if } \left(\frac{\Delta}{p}\right) = -1, \end{cases}$$

where I is the 2×2 identity matrix and $\left(\frac{\Delta}{p}\right)$ denotes the Legendre symbol.

Proof: Let $u_{-1} = 0$, $u_0 = 1$, and $u_{n+1} = (a + d)u_n - (ad - bc)u_{n-1}$ ($n = 0, 1, 2, \dots$). Then $u_n = u_n(-a - d, ad - bc)$. Since the characteristic polynomial of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $x^2 - (a + d)x + ad - bc$, using Theorem 3.2 we see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = u_{n-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + (u_n - (a + d)u_{n-1}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u_n - du_{n-1} & bu_{n-1} \\ cu_{n-1} & u_n - au_{n-1} \end{pmatrix}. \quad (3.3)$$

Clearly, $\Delta = (a + d)^2 - 4(ad - bc)$. Thus, by [10, pp. 46-47],

$$u_{p-1 - \left(\frac{\Delta}{p}\right)} \equiv 0 \pmod{p}, \quad u_{p-1} \equiv \left(\frac{\Delta}{p}\right) \pmod{p}.$$

Putting the above together yields

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p - \left(\frac{\Delta}{p}\right)} \equiv u_{p - \left(\frac{\Delta}{p}\right)} I \pmod{p}.$$

If $\left(\frac{\Delta}{p}\right) = 1$ then $u_{p - \left(\frac{\Delta}{p}\right)} = u_{p-1} \equiv \left(\frac{\Delta}{p}\right) = 1 \pmod{p}$. If $\left(\frac{\Delta}{p}\right) = -1$, then $u_{p-1} \equiv -1 \pmod{p}$ and $u_p \equiv 0 \pmod{p}$. Thus, $u_{p - \left(\frac{\Delta}{p}\right)} = u_{p+1} = (a + d)u_p - (ad - bc)u_{p-1} \equiv ad - bc \pmod{p}$.

If $\left(\frac{\Delta}{p}\right) = 0$, then $p \mid \Delta$. Using Fermat's little theorem, we see that

$$\begin{aligned} u_{p-\left(\frac{\Delta}{p}\right)} &= u_p = \frac{1}{\sqrt{\Delta}} \left\{ \left(\frac{a+d+\sqrt{\Delta}}{2} \right)^{p+1} - \left(\frac{a+d-\sqrt{\Delta}}{2} \right)^{p+1} \right\} \\ &= \frac{2}{2^{p+1}} \sum_{2 \mid k} \binom{p+1}{k} (a+d)^{p+1-k} (\sqrt{\Delta})^{k-1} \\ &\equiv \frac{2}{2^{p+1}} (p+1)(a+d)^p \equiv \frac{a+d}{2} \pmod{p}. \end{aligned}$$

Combining the above produces the desired result.

4. AN IDENTITY FOR $\{u_n(a_1, \dots, a_m)\}$

Using Theorems 3.1 and 3.2, one can prove the following identity.

Theorem 4.1: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0$, $a_0 = 1$, and $u_n = u_n(a_1, \dots, a_m)$. Then, for $n, l \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$, we have

$$u_{kn+l} = \sum_{k_0+k_1+\dots+k_{m-1}=k} \frac{k!}{k_0!k_1!\dots k_{m-1}!} \prod_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right)^{k_r} u_{m-1}^{\sum_{r=0}^{m-1} r k_r + l}.$$

Proof: Let A, D , and M_n denote the matrices as in the proof of Theorem 3.1. It is clear that the characteristic polynomial of A is $x^m + a_1 x^{m-1} + \dots + a_m$. So, by Theorem 3.2,

$$A^n = \sum_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right) A^r.$$

From this and the multinomial theorem for square matrices, it follows that

$$\begin{aligned} A^{kn+l} &= \left(\sum_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right) A^r \right)^k A^l \\ &= \sum_{k_0+k_1+\dots+k_{m-1}=k} \frac{k!}{k_0!k_1!\dots k_{m-1}!} \prod_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right)^{k_r} A^{\sum_{r=0}^{m-1} r k_r + l}. \end{aligned}$$

Multiplying both sides on the left by D^{-1} and then applying Theorem 3.1, we see that

$$M_{kn+l} = \sum_{k_0+k_1+\dots+k_{m-1}=k} \frac{k!}{k_0!k_1!\dots k_{m-1}!} \prod_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right)^{k_r} M_{m-1}^{\sum_{r=0}^{m-1} r k_r + l}.$$

Now, comparing the elements in row 1 and column 1 of the matrices on both sides yields the result.

Corollary 4.1: Let a_1 and a_2 be complex numbers with $a_2 \neq 0$. If $\{U_r\}$ is the Lucas sequence given by $U_0 = 0$, $U_1 = 1$, and $U_r + a_1 U_{r-1} + a_2 U_{r-2} = 0$ ($r = 0, \pm 1, \pm 2, \dots$), then

$$U_{kn+l} = \sum_{i=0}^k \binom{k}{i} (-a_2 U_{n-1})^{k-i} U_n^i U_{l+i}, \tag{4.1}$$

where $n, l \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$.

Proof: Note that $U_r = u_{r-1}(a_1, a_2)$. By taking $m = 2$ in Theorem 4.1 and then replacing l by $l - 1$, we obtain the result.

Remark 4.1: When $n, l \geq 0$ and $a_1 = a_2 = -1$, the result of Corollary 4.1 was established by my brother Zhi-Wei Sun [14]. (In the case $l = 0$, the result is due to Siebeck [2, p. 394].) Here I give the following general identity,

$$U_s^k U'_{kn+l} = \sum_{i=0}^k \binom{k}{i} U_n^i (-a_2^s U_{n-s})^{k-i} U'_{l+is}, \tag{4.2}$$

where $\{U'_r\}$ satisfies the recurrence relation $U'_r + a_1 U'_{r-1} + a_2 U'_{r-2} = 0$ ($n = 0, \pm 1, \pm 2, \dots$). This can be proved easily by using the relation $U'_r = U'_1 U_r - a_2 U'_0 U_{r-1}$ and the known formula

$$U_r = \frac{1}{\sqrt{a_1^2 - 4a_2}} \left\{ \left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \right)^r - \left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} \right)^r \right\}.$$

Corollary 4.2: Let a_1, \dots, a_m be complex numbers with $a_m \neq 0, a_0 = 1$, and $u_n = u_n(a_1, \dots, a_m)$. For $n, l \in \mathbb{Z}$, we have

$$u_{n+l} = \sum_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right) u_{r+l}.$$

Proof: Putting $k = 1$ in Theorem 4.1 yields the result.

Corollary 4.3: Let p be a prime, $a_1, \dots, a_m \in \mathbb{Z}, p \nmid a_m, l, n \in \mathbb{Z}, a_0 = 1$, and $u_n = u_n(a_1, \dots, a_m)$. Then

$$u_{np+l} \equiv \sum_{r=0}^{m-1} \sum_{s=r}^{m-1} a_{s-r} u_{n-s} u_{rp+l} \pmod{p}.$$

Proof: If $k_0 + \dots + k_{m-1} = p$, then

$$\frac{p!}{k_0! \dots k_{m-1}!} \equiv \begin{cases} 1 \pmod{p} & \text{if } p = k_r, \text{ for some } r \in \{0, \dots, m-1\}, \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

This, together with Theorem 4.1 and Fermat's little theorem, gives

$$\begin{aligned} u_{np+l} &\equiv \sum_{r=0}^{m-1} \left(\sum_{s=r}^{m-1} a_{s-r} u_{n-s} \right)^p u_{rp+l} \\ &\equiv \sum_{r=0}^{m-1} \sum_{s=r}^{m-1} a_{s-r} u_{n-s} u_{rp+l} \pmod{p}, \end{aligned}$$

which is the result.

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F_{81839} Is Prime

David Broadhurst and Bouk de Water have recently proved that F_{81839} is prime.