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# The expansion of $\prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk})$

by

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**1. Introduction.** Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be the sets of positive integers, integers and real numbers respectively. A negative integer $d$ with $d \equiv 0,1 \pmod 4$ is called a *discriminant*. The *conductor* of the discriminant $d$ is the largest positive integer $f = f(d)$ such that $d/f^2 \equiv 0,1 \pmod 4$.

For integers $a, b$ and $c$ with $a, c > 0$ and $b^2 - 4ac < 0$, we use $(a, b, c)$ to denote the form $ax^2 + bxy + cy^2$. Two forms $(a, b, c)$ and $(a', b', c')$ are *equivalent* ($(a, b, c) \sim (a', b', c')$) if there exist integers $\alpha, \beta, \gamma$ and $\delta$ with $\alpha\delta - \beta\gamma = 1$ such that the substitution $x = \alpha X + \beta Y,\ y = \gamma X + \delta Y$ transforms $(a, b, c)$ to $(a', b', c')$. The substitutions $x = Y,\ y = -X$ and $x = X + kY,\ y = Y$ imply

$$(a, b, c) \sim (c, -b, a) \sim (a, 2ak + b, ak^2 + bk + c) \quad (k \in \mathbb{Z}) \tag{1.1}$$

(see also [D, p. 141]). We denote the equivalence class of $(a, b, c)$ by $[a, b, c]$, and the form class group of discriminant $d$ by $H(d)$.

Let $\mathbb{Z}^2 = \{\langle x, y\rangle : x, y \in \mathbb{Z}\}$. For $n \in \mathbb{N}$ and $a, b, c \in \mathbb{Z}$ with $a, c > 0$ and $b^2 - 4ac < 0$, we define

$$R(a, b, c; n) = |\{\langle x, y\rangle \in \mathbb{Z}^2 : n = ax^2 + bxy + cy^2\}|. \tag{1.2}$$

If $(a, b, c) \sim (a', b', c')$, it is known that (see [SW1])

$$R(a, b, c; n) = R(a, -b, c; n) = R(a', b', c'; n). \tag{1.3}$$

In this paper we extend some results in [SW2]. In particular, we show that for $a, b \in \mathbb{N}$ and $q \in \mathbb{R}$ with $|q| < 1$,

$$\prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk}) = 1 + \sum_{n=1}^{\infty}\frac{1}{2}\big(R(a+b, 12(a-b), 36(a+b); 24n+a+b) - R(4(a+b), 12(a-b), 9(a+b); 24n+a+b)\big)q^n.$$

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In the special case $a+b=24$, the result is equivalent to Theorem 2.2 of [SW2].

For $n\in\mathbb{N}$ and $b\in\{1,2,5\}$, in Section 4, we determine the number of representations of $n$ as

$$n=\frac{3x^2-x}{2}+b\,\frac{3y^2-y}{2}.$$

For example, we have

$$\left|\left\{\langle x,y\rangle\in\mathbb{Z}^2 : n=\frac{3x^2-x}{2}+\frac{3y^2-y}{2}\right\}\right|=\sum_{k\mid(12n+1)}(-1)^{(k-1)/2}.$$

In Section 5 we show that if $k,m,n\in\mathbb{N}$ with $2\nmid k$, $2\mid n$, $m<20k$ and $n>9k$, then

$$R(k,2m,20m;n)=R(k+4m,18m,20m;n).$$

In addition to the above notation, we also use throughout this paper the following notation: $\mathrm{ord}_p n$ denotes the nonnegative integer $\alpha$ such that $p^\alpha\,\|\,n$ (that is, $p^\alpha\mid n$ but $p^{\alpha+1}\nmid n$), $(a,b)$ is the greatest common divisor of the integers $a$ and $b$ (not both zero), and $\left(\frac{a}{m}\right)$ is the Legendre–Jacobi–Kronecker symbol.

## 2. General formulas for $f_{a,b}(r,m;n)$ and $R_{a,b}(r,m;n)$

Definition 2.1. For $r,m\in\mathbb{N}$ and $q\in\mathbb{R}$, we define

$$f(r,m;q)=\prod_{n=0}^{\infty}\{(1-q^{mn+(m-r)/2})(1-q^{mn+m})(1-q^{mn+(m+r)/2})\}\quad(|q|<1).$$

From Jacobi's triple product identity (cf. [HW, Theorem 352, p. 282 (with $x=q^{m/2},\ z=-q^{-r/2}$)]) we know that

$$f(r,m;q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(mn^2-rn)/2}\quad(|q|<1). \tag{2.1}$$

In particular,

$$f(1,3;q)=\prod_{n=1}^{\infty}(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2}\quad(|q|<1).$$

This is Euler's pentagonal number theorem.

Definition 2.2. For $a,b,r,m\in\mathbb{N}$ with $2\mid(a,b)(m-r)$, we define $f_{a,b}(r,m;n)$ by

$$f(r,m;q^a)f(r,m;q^b)=1+\sum_{n=1}^{\infty}f_{a,b}(r,m;n)q^n$$

and $R_{a,b}(r,m;n)$ by

$$R_{a,b}(r,m;n)=\left|\left\{\langle x,y\rangle\in\mathbb{Z}^2: n=a\,\frac{mx^2-rx}{2}+b\,\frac{my^2-ry}{2}\right\}\right|.$$

Clearly

$$\Big(\sum_{n=-\infty}^{\infty} q^{a(mn^2-rn)/2}\Big)\Big(\sum_{n=-\infty}^{\infty} q^{b(mn^2-rn)/2}\Big) = 1+\sum_{n=1}^{\infty} R_{a,b}(r,m;n)q^n \qquad (|q|<1).$$

THEOREM 2.1. *Let* $a,b,r,m,n\in\mathbb{N}$ *with* $2\mid(a,b)(m-r)$. *Then*

$$f_{a,b}(r,m;n) = \begin{cases} 2\sum\limits_{\substack{x,y\in\mathbb{Z},\ x\equiv r/2\ (\mathrm{mod}\ m)\\ (a+b)x^2+4amxy+4am^2y^2=2mn+(a+b)r^2/4}} 1 - \sum\limits_{\substack{x,y\in\mathbb{Z},\ x\equiv r/2\ (\mathrm{mod}\ m)\\ (a+b)x^2+2amxy+am^2y^2=2mn+(a+b)r^2/4}} 1 & \text{if } 2\mid r,\\ \sum\limits_{\substack{x,y\in\mathbb{Z},\ x\equiv r\ (\mathrm{mod}\ 2m)\\ (a+b)x^2+8amxy+16am^2y^2=8mn+(a+b)r^2}} 1 - \sum\limits_{\substack{x,y\in\mathbb{Z},\ x\equiv r\ (\mathrm{mod}\ 2m)\\ (a(2m-1)^2+b)x^2+8am(1-2m)xy+16am^2y^2=8mn+(a+b)r^2}} 1 & \text{if } 2\nmid r \end{cases}$$

*and*

$$R_{a,b}(r,m;n) = \begin{cases} \sum\limits_{\substack{x,y\in\mathbb{Z},\ x\equiv r/2\ (\mathrm{mod}\ m)\\ (a+b)x^2+2amxy+am^2y^2=2mn+(a+b)r^2/4}} 1 & \text{if } 2\mid r,\\ \sum\limits_{\substack{x,y\in\mathbb{Z},\ x\equiv r\ (\mathrm{mod}\ 2m)\\ (a+b)x^2+4amxy+4am^2y^2=8mn+(a+b)r^2}} 1 & \text{if } 2\nmid r. \end{cases}$$

*Proof.* For $c\in\{0,1\}$, we see that

$$\begin{aligned}\Big(\sum_{n=-\infty}^{\infty}(-1)^{cn}q^{a(mn^2-rn)/2}\Big)\Big(\sum_{n=-\infty}^{\infty}(-1)^{cn}q^{b(mn^2-rn)/2}\Big) &= \sum_{n=0}^{\infty}\sum_{\substack{x,y\in\mathbb{Z}\\ a(mx^2-rx)/2+b(my^2-ry)/2=n}}(-1)^{c(x+y)}q^n\\ &= \sum_{n=0}^{\infty}\sum_{\substack{x,y\in\mathbb{Z}\\ a(mx^2+rx)/2+b(my^2+ry)/2=n}}(-1)^{c(x-y)}q^n\end{aligned}$$

$$=\sum_{n=0}^{\infty}\sum_{\substack{x,y\in\mathbb{Z}\\ a(4m^2x^2+4mrx)+b(4m^2y^2+4mry)=8mn}}(-1)^{c(x-y)}q^n$$

$$=\sum_{n=0}^{\infty}\sum_{\substack{x,y\in\mathbb{Z}\\ a(2mx+r)^2+b(2my+r)^2=8mn+(a+b)r^2}}(-1)^{c(x-y)}q^n$$

$$=\sum_{n=0}^{\infty}\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv y\equiv r\pmod{2m}\\ ay^2+bx^2=8mn+(a+b)r^2}}(-1)^{c(x-y)/(2m)}q^n$$

$$=\sum_{n=0}^{\infty}\sum_{\substack{x,z\in\mathbb{Z},\ x\equiv r\pmod{2m}\\ a(x+2mz)^2+bx^2=8mn+(a+b)r^2}}(-1)^{cz}q^n.$$

Thus, by (2.1) and Definition 2.2 we have

$$f_{a,b}(r,m;n)=\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv r\pmod{2m}\\ a(x+2my)^2+bx^2=8mn+(a+b)r^2}}(-1)^y \tag{2.2}$$

and

$$R_{a,b}(r,m;n)=\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv r\pmod{2m}\\ a(x+2my)^2+bx^2=8mn+(a+b)r^2}}1. \tag{2.3}$$

If $2\mid r$, then

$$\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv r\pmod{2m}\\ a(x+2my)^2+bx^2=8mn+(a+b)r^2}}(-1)^{cy}=\sum_{\substack{x,y\in\mathbb{Z},\ 2x\equiv r\pmod{2m}\\ a(2x+2my)^2+b(2x)^2=8mn+(a+b)r^2}}(-1)^{cy}$$

$$=\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv r/2\pmod{m}\\ a(x+my)^2+bx^2=2mn+(a+b)r^2/4}}(-1)^{cy}$$

and

$$\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv r/2\pmod{m}\\ a(x+my)^2+bx^2=2mn+(a+b)r^2/4}}(-1)^y$$

$$=\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv r/2\pmod{m}\\ a(x+my)^2+bx^2=2mn+(a+b)r^2/4}}(1+(-1)^y)-\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv r/2\pmod{m}\\ a(x+my)^2+bx^2=2mn+(a+b)r^2/4}}1$$

$$=\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv r/2\pmod{m}\\ a(x+2my)^2+bx^2=2mn+(a+b)r^2/4}}2-\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv r/2\pmod{m}\\ a(x+my)^2+bx^2=2mn+(a+b)r^2/4}}1$$

$$= 2\sum_{\substack{x,y\in\mathbb{Z},\, x\equiv r/2\ (\mathrm{mod}\, m)\\ (a+b)x^2+4amxy+4am^2y^2=2mn+(a+b)r^2/4}} 1 \quad - \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv r/2\ (\mathrm{mod}\, m)\\ (a+b)x^2+2amxy+am^2y^2=2mn+(a+b)r^2/4}} 1.$$

Thus, we see that the result holds when $2\mid r$.

If $2\nmid r$, then

$$\sum_{\substack{x,y\in\mathbb{Z},\, x\equiv r\ (\mathrm{mod}\, 2m)\\ a(x+2my)^2+bx^2=8mn+(a+b)r^2}} (-1)^y$$

$$= \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv r\ (\mathrm{mod}\, 2m),\, 2\mid y\\ a(x+2my)^2+bx^2=8mn+(a+b)r^2}} 1 - \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv r\ (\mathrm{mod}\, 2m),\, 2\mid y-x\\ a(x+2my)^2+bx^2=8mn+(a+b)r^2}} 1$$

$$= \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv r\ (\mathrm{mod}\, 2m)\\ a(x+4my)^2+bx^2=8mn+(a+b)r^2}} 1 - \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv r\ (\mathrm{mod}\, 2m)\\ a(x+2m(2y-x))^2+bx^2=8mn+(a+b)r^2}} 1$$

$$= \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv r\ (\mathrm{mod}\, 2m)\\ (a+b)x^2+8amxy+16am^2y^2=8mn+(a+b)r^2}} 1 \quad - \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv r\ (\mathrm{mod}\, 2m)\\ (a(2m-1)^2+b)x^2+8am(1-2m)xy+16am^2y^2=8mn+(a+b)r^2}} 1.$$

This together with (2.2) and (2.3) yields the result in the case $2\nmid r$. The proof is now complete.

We note that Theorem 2.1 can be viewed as a generalization of [SW2, Proposition 2.1].

## 3. The expansion of $\prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk})$

LEMMA 3.1 ([SW2, p. 356]). *Let* $n\in\mathbb{N}$ *and* $a,b,c\in\mathbb{Z}$ *with* $a,c>0$ *and* $b^2-4ac<0$. *Then*

$$2\sum_{\substack{x,y\in\mathbb{Z},\, x\equiv 1\ (\mathrm{mod}\, 6)\\ ax^2+bxy+cy^2=n}} 1 = R(a,b,c;n) - R(9a,3b,c;n) - R(4a,2b,c;n) + R(36a,6b,c;n).$$

Theorem 3.1. *Let $a,b\in\mathbb{N}$ and $q\in\mathbb{R}$ with $|q|<1$. Then*

$$\prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk})$$
$$=1+\sum_{n=1}^{\infty}\frac12\big(R(a+b,12(a-b),36(a+b);24n+a+b)$$
$$-R(4(a+b),12(a-b),9(a+b);24n+a+b)\big)q^n.$$

*Proof.* Let $n\in\mathbb{N}$ and $n'=24n+a+b$. Taking $m=3$ and $r=1$ in Theorem 2.1 and then applying Lemma 3.1, we have

$$f_{a,b}(1,3;n)$$
$$=\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv1\ (\mathrm{mod}\ 6)\\ (a+b)x^2+24axy+144ay^2=n'}}1-\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv1\ (\mathrm{mod}\ 6)\\ (25a+b)x^2-120axy+144ay^2=n'}}1$$
$$=\frac12\big(R(a+b,24a,144a;n')-R(9(a+b),72a,144a;n')$$
$$-R(4(a+b),48a,144a;n')+R(36(a+b),144a,144a;n')\big)$$
$$-\frac12\big(R(25a+b,-120a,144a;n')-R(9(25a+b),-360a,144a;n')$$
$$-R(4(25a+b),-240a,144a;n')+R(36(25a+b),-720a,144a;n')\big).$$

For $a',b',c',k\in\mathbb{Z}$ we see that

$$(a',b',c')\sim(c',-b',a')\sim(c',-2c'k-b',k^2c'+kb'+a')$$
$$\sim(a'+kb'+k^2c',b'+2kc',c').$$

Thus $(9(25a+b),-360a,144a)\sim(9(a+b),-72a,144a)$, $(4(25a+b),-240a,144a)\sim(4(a+b),48a,144a)$ and $(36(25a+b),-720a,144a)\sim(36(a+b),144a,144a)$. Hence using (1.3) we see that

$$R(9(25a+b),-360a,144a;n')=R(9(a+b),72a,144a;n'),$$
$$R(4(25a+b),-240a,144a;n')=R(4(a+b),48a,144a;n'),$$
$$R(36(25a+b),-720a,144a;n')=R(36(a+b),144a,144a;n').$$

Now combining the above we obtain

$$(3.1)\qquad 2f_{a,b}(1,3;n)=R(a+b,24a,144a;24n+a+b)$$
$$-R(25a+b,-120a,144a;24n+a+b).$$

From (1.1) we have

$$(a+b,24a,144a)\sim(a+b,-12(a+b)+24a,36(a+b)-6\cdot24a+144a)$$
$$\sim(a+b,12(a-b),36(a+b))$$

and

$$\begin{aligned}&(25a+b,-120a,144a)\\ &\qquad\sim(25a+b,4(25a+b)-120a,4(25a+b)-2\cdot120a+144a)\\ &\qquad\sim(25a+b,4b-20a,4(a+b))\sim(4(a+b),20a-4b,25a+b)\\ &\qquad\sim(4(a+b),-8(a+b)+20a-4b,4(a+b)-(20a-4b)+25a+b)\\ &\qquad\sim(4(a+b),12(a-b),9(a+b)).\end{aligned}$$

Thus applying the above and (1.3) we get

$$\begin{aligned}2f_{a,b}(1,3;n)&=R(a+b,12(a-b),36(a+b);24n+a+b)\\ &\qquad-R(4(a+b),12(a-b),9(a+b);24n+a+b).\end{aligned}$$

To see the result, we note that

$$\prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk})=1+\sum_{n=1}^{\infty}f_{a,b}(1,3;n)q^n. \tag{3.2}$$

Corollary 3.1. *For* $a,b,n\in\mathbb{N}$ *we have*

$$\begin{aligned}&R(a+b,12(a-b),36(a+b);24n+a+b)\\ &\qquad\equiv R(4(a+b),12(a-b),9(a+b);24n+a+b)\pmod 2.\end{aligned}$$

Corollary 3.2. *Let* $k,m,n\in\mathbb{N}$ *with* $m<12k$ *and* $2(m,12k)\nmid n$. *Then*

$$R(k,2m,12m;n+k)=R(k+2m,10m,12m;n+k).$$

*Proof.* Set $a=2m$ and $b=24k-2m$. Then $(a,b)=(2m,24k)=2(m,12k)$ and so $(a,b)\nmid n$. Hence, by (3.2) we have $f_{a,b}(1,3;n)=0$. This together with (3.1) gives

$$R(a+b,24a,144a;24n+a+b)=R(25a+b,120a,144a;24n+a+b).$$

That is,

$$R(24k,48m,288m;24(n+k))=R(24k+48m,240m,288m;24(n+k)).$$

This yields the result.

Remark 3.1. Let $m\in\{1,2,\dots,11\}$. Taking $k=1$ in Corollary 3.2 we see that if $2\mid n$, then $R(1,2m,12m;n)=R(2m+1,10m,12m;n)$. As $(1,2m,12m)\sim(1,0,m(12-m))$ and $(2m+1,10m,12m)\sim(2m+1,2m-4,4)\sim(4,4-2m,2m+1)$, we deduce that

$$R(1,0,m(12-m);n)=R(4,4-2m,2m+1;n). \tag{3.3}$$

When $m\le6$, (3.3) has been given in [SW2, Corollary 2.1]. For $m\le5$ see also [KW].

Lemma 3.2. *Let* $a,b\in\mathbb{N}$. *Then the form* $(a+b,12(a-b),36(a+b))$ *is not equivalent to* $(4(a+b),\pm12(a-b),9(a+b))$.

*Proof.* If $(a+b,12(a-b),36(a+b)) \not\sim (4(a+b),\pm 12(a-b),9(a+b))$, then clearly $(a+b,12(b-a),36(a+b)) \not\sim (4(a+b),\pm 12(a-b),9(a+b))$. Thus we only need to consider the case $a \le b$. Now we assume $a \le b$. Let

$$F_1 = \begin{cases} (144a,-24a,a+b) & \text{if } b > 143a, \\ (a+b,24a,144a) & \text{if } 23a \le b \le 143a, \\ (a+b,22a-2b,121a+b) & \text{if } 7a \le b < 23a, \\ (a+b,20a-4b,100a+4b) & \text{if } \frac{19}{5}a \le b < 7a, \\ (a+b,18a-6b,81a+9b) & \text{if } \frac{17}{7}a \le b < \frac{19}{5}a, \\ (a+b,16a-8b,64a+16b) & \text{if } \frac{5}{3}a \le b < \frac{17}{7}a, \\ (a+b,14a-10b,49a+25b) & \text{if } \frac{13}{11}a \le b < \frac{5}{3}a, \\ (a+b,12(a-b),36(a+b)) & \text{if } b < \frac{13}{11}a. \end{cases}$$

It is easily seen that $F_1$ is a reduced form of discriminant $-576ab$. Using (1.1) we see that $(a+b,12(a-b),36(a+b))$ is equivalent to $F_1$. Set

$$F_2 = \begin{cases} (144a,120a,25a+b) & \text{if } b \ge 119a, \\ (25a+b,-120a,144a) & \text{if } 95a < b < 119a, \\ (25a+b,2b-70a,49a+b) & \text{if } 15a < b \le 95a, \\ (25a+b,4b-20a,4a+4b) & \text{if } 7a \le b \le 15a, \\ (4a+4b,20a-4b,25a+b) & \text{if } 2a \le b < 7a, \\ (4(a+b),12(a-b),9(a+b)) & \text{if } b < 2a. \end{cases}$$

It is easily seen that $F_2$ is also a reduced form of discriminant $-576ab$. Using (1.1) we see that $(4(a+b),12(a-b),9(a+b))$ is equivalent to $F_2$. Clearly $F_1$ is different from $F_2$ and from the converse of $F_2$, and hence not equivalent to them, by Lagrange's theorem for reduced forms. Putting all the above together we obtain the result.

THEOREM 3.2. *For $a,b \in \mathbb{N}$ let $\{c_n\}$ be given by*

$$\prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk}) = 1+\sum_{n=1}^{\infty} c_n q^n \quad (|q|<1).$$

(i) *Assume $a=b$, $a \mid n$ and $p = 12n/a+1$. If $p$ is a prime, then*

$$c_n = \begin{cases} 2 & \text{if } p \text{ is represented by } x^2+36y^2, \\ -2 & \text{if } p \text{ is represented by } 4x^2+9y^2, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Assume $a \ne b$, $d = (a+b,24(a,b))$ and $p = (24n+a+b)/d$. If $p$ is a prime, then*

$$c_n = \begin{cases} 1 & \textit{if } p \textit{ is represented by } \frac{a+b}{d}x^2 + \frac{12(a-b)}{d}xy + \frac{36(a+b)}{d}y^2, \\ -1 & \textit{if } p \textit{ is represented by } \frac{4(a+b)}{d}x^2 + \frac{12(a-b)}{d}xy + \frac{9(a+b)}{d}y^2, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* When $a = b$ we set $d = 2a$. As

$$\begin{aligned}(a+b, 12(a-b)) &= (a+b, 24a - 12(a+b)) = (a+b, 24a) \\ &= (a,b)((a+b)/(a,b), 24a/(a,b)) \\ &= (a,b)((a+b)/(a,b), 24) = d,\end{aligned}$$

we see that $((a+b)/d, 12(a-b)/d, 36(a+b)/d)$ and $(4(a+b)/d, 12(a-b)/d, 9(a+b)/d)$ are primitive forms of discriminant $-576ab/d^2$. Set $d_1 = ((a+b)/(a,b), 24)$. Then $d = (a,b)d_1$ and $d_1 \mid 24$. If $d_1 \in \{1,2,3,4\}$, then

$$576\,\frac{ab}{d^2} = \frac{576}{d_1^2} \cdot \frac{a}{(a,b)} \cdot \frac{b}{(a,b)} \geq \frac{576}{4^2} > 4.$$

If $d_1 \in \{6,8,12,24\}$, then

$$\begin{aligned}\frac{576ab}{d^2} &= \frac{576}{d_1^2} \cdot \frac{a}{(a,b)} \cdot \frac{b}{(a,b)} \geq \frac{576}{d_1^2}\left(\frac{a}{(a,b)} + \frac{b}{(a,b)} - 1\right) \\ &\geq \frac{576}{d_1^2}\,(d_1 - 1) \geq d_1 - 1 > 4.\end{aligned}$$

Thus we always have $-576ab/d^2 < -4$. From Theorem 3.1 we have

$$\begin{aligned}c_n &= \frac{1}{2}\big(R(a+b, 12(a-b), 36(a+b); 24n+a+b) \\ &\quad - R(4(a+b), 12(a-b), 9(a+b); 24n+a+b)\big) \\ &= \frac{1}{2}\left(R\Big(\frac{a+b}{d}, \frac{12(a-b)}{d}, \frac{36(a+b)}{d}; p\Big)\right. \\ &\quad \left. - R\Big(\frac{4(a+b)}{d}, \frac{12(a-b)}{d}, \frac{9(a+b)}{d}; p\Big)\right).\end{aligned}$$

If $p$ is not represented by $((a+b)/d, 12(a-b)/d, 36(a+b)/d)$ and $(4(a+b)/d, 12(a-b)/d, 9(a+b)/d)$, by the above we have $c_n = 0$. If $p$ is represented by $((a+b)/d, 12(a-b)/d, 36(a+b)/d)$, by [SW1, Lemma 5.2] and Lemma 3.2 we have

$$R\Big(\frac{a+b}{d}, \frac{12(a-b)}{d}, \frac{36(a+b)}{d}; p\Big) = \begin{cases} 2 & \text{if } a \neq b, \\ 4 & \text{if } a = b \end{cases}$$

and

$$R\Big(\frac{4(a+b)}{d}, \frac{12(a-b)}{d}, \frac{9(a+b)}{d}; p\Big) = 0.$$

Hence $c_n = 1$ or 2 according as $a \neq b$ or $a = b$. Similarly, if $p$ is represented by $(4(a+b)/d, 12(a-b)/d, 9(a+b)/d)$, by [SW1, Lemma 5.2], Lemma 3.2

and the above we have $c_n = -1$ or $-2$ according as $a \neq b$ or $a = b$. This concludes the proof.

For example, let

$$\prod_{k=1}^{\infty}(1-q^{2k})(1-q^{3k}) = 1 + \sum_{n=1}^{\infty} c_n q^n \quad (|q|<1).$$

If $p = 24n+5$ is a prime, taking $a = 3$ and $b = 2$ in Theorem 3.2 we have

$$c_n = \begin{cases} 1 & \text{if } p \text{ is represented by } 5x^2+12xy+180y^2, \\ -1 & \text{if } p \text{ is represented by } 20x^2+12xy+45y^2, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 3.2. Let

$$q\prod_{k=1}^{\infty}(1-q^{12k})^2 = \sum_{n=1}^{\infty}\phi_{12}(n)q^n \quad (|q|<1).$$

The value of $\phi_{12}(n)$ has been given in [SW2, Theorem 4.5(iv)]. It is easily seen that

$$\prod_{k=1}^{\infty}(1-q^{ak})^2 = 1 + \sum_{m=1}^{\infty}\phi_{12}(12m+1)q^{am} \quad (|q|<1).$$

For $a, b, n \in \mathbb{N}$ let

$$(3.4) \qquad \phi(a,b;n) = \frac{1}{2}\big(R(a+b, 12(a-b), 36(a+b); 24n+a+b) \\ - R(4(a+b), 12(a-b), 9(a+b); 24n+a+b)\big).$$

From Corollary 3.1 we know that $\phi(a,b;n) \in \mathbb{Z}$. For a rational number $m$ we let

$$\sigma(m) = \begin{cases} \sum_{d\mid m} d & \text{if } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\{a_n\}$ and $\{b_n\}$ are two sequences satisfying

$$a_1 = b_1 \quad \text{and} \quad b_n + a_1 b_{n-1} + \cdots + a_{n-1}b_1 = na_n \quad (n = 2, 3, \ldots),$$

we say that $(a_n, b_n)$ is a *Newton–Euler pair* as in [S]. Now we state the following result.

THEOREM 3.3. *Let $a, b \in \mathbb{N}$. Then $(\phi(a,b;n), -a\sigma(n/a) - b\sigma(n/b))$ is a Newton–Euler pair. That is, for $n \in \mathbb{N}$,*

$$a\sigma\Big(\frac{n}{a}\Big) + b\sigma\Big(\frac{n}{b}\Big) + \sum_{k=1}^{n-1}\Big(a\sigma\Big(\frac{k}{a}\Big) + b\sigma\Big(\frac{k}{b}\Big)\Big)\phi(a,b;n-k) = -n\phi(a,b;n).$$

*Proof.* Suppose $q \in \mathbb{R}$ and $|q| < 1$. As

$$1 - q^n = \prod_{r=0}^{n-1}(1 - e^{2\pi i r/n}q),$$

applying Theorem 3.1 we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \phi(a,b;n)q^n &= \prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk}) \\ &= \prod_{k=1}^{\infty} \prod_{r=0}^{ak-1}(1 - e^{2\pi i r/(ak)}q) \prod_{s=0}^{bk-1}(1 - e^{2\pi i s/(bk)}q). \end{aligned}$$

Observe that

$$\begin{aligned} &\sum_{k=1}^{\infty}\Big\{ \sum_{r=0}^{ak-1}(e^{2\pi i r/(ak)})^n + \sum_{s=0}^{bk-1}(e^{2\pi i s/(bk)})^n \Big\} \\ &\qquad = \sum_{\substack{k\in\mathbb{N} \\ ak\mid n}} ak + \sum_{\substack{k\in\mathbb{N} \\ bk\mid n}} bk = a\sigma(n/a) + b\sigma(n/b). \end{aligned}$$

From the above and [S, Example 1, p. 103] we deduce the result.

THEOREM 3.4. *Let* $a, b, n \in \mathbb{N}$. *Then*

$$\phi(a,b;n) = \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+\cdots+k_n} \frac{(a\sigma(1/a)+b\sigma(1/b))^{k_1}\cdots(a\sigma(n/a)+b\sigma(n/b))^{k_n}}{1^{k_1}\cdot k_1!\cdots n^{k_n}\cdot k_n!}.$$

*Proof.* This is immediate from Theorem 3.3 and [S, Theorem 2.2].

REMARK 3.3. For $a \in \{1, 2, \ldots, 12\}$ and $n \in \mathbb{N}$, by (3.4) we have

$$\begin{aligned} 2\phi(a, 24-a; n) &= R(24, 12(2a-24), 36\cdot 24; 24n+24) \\ &\quad - R(4\cdot 24, 12(2a-24), 9\cdot 24; 24n+24) \\ &= R(1, a-12, 36; n+1) - R(4, a-12, 9; n+1). \end{aligned}$$

Hence, for $n > 1$,

$$\begin{aligned} &2\phi(a, 24-a; n-1) \\ &\qquad = R(1, 12-a, 36; n) - R(4, 12-a, 9; n) \\ &\qquad = R\Big(1, \frac{1}{2}(1-(-1)^a), \frac{1}{4}\Big(24a - a^2 + \frac{1}{2}(1-(-1)^a)\Big); n\Big) \\ &\qquad\quad - R(4, 4-a, a+1; n). \end{aligned}$$

Suppose $\phi(a, 24-a; 0)=1$. Using [SW2, Theorems 2.2, 7.2 and 8.2] we see that $\phi(a, 24-a; n-1)$ is a multiplicative function of $n$ for $a \in \{1, 2, 3, 4, 6, 8, 12\}$.

The values of $\phi(a,24-a;n-1)$ ($a\in\{1,2,3,4,6,8,12\}$) have been given in [SW2, Theorems 4.4 and 4.5].

## 4. Formulas for $R_{1,1}(1,3;n), R_{1,2}(1,3;n)$ and $R_{1,5}(1,3;n)$

THEOREM 4.1. *Let $a,b,n\in\mathbb{N}$. Then*

$$\begin{aligned}2R_{a,b}(1,3;n)=R(a+b,12a,36a;24n+a+b)\\ -R(9(a+b),36a,36a;24n+a+b)\\ -R(4(a+b),24a,36a;24n+a+b)\\ +R(36(a+b),72a,36a;24n+a+b).\end{aligned}$$

*Proof.* From Theorem 2.1 we have

$$R_{a,b}(1,3;n)=\sum_{\substack{x,y\in\mathbb{Z},\ x\equiv 1\pmod 6\\ (a+b)x^2+12axy+36ay^2=24n+a+b}}1.$$

Now applying Lemma 3.1 we obtain the result.

COROLLARY 4.1. *If $a,b,n\in\mathbb{N}$, $3\nmid(a+b)$ and $4\nmid(a+b)$, then*

$$R_{a,b}(1,3;n)=\frac12 R(a+b,6(a-b),9(a+b);24n+a+b).$$

*Proof.* From Theorem 4.1 we have

$$2R_{a,b}(1,3;n)=R(a+b,12a,36a;24n+a+b).$$

Note that

$$\begin{aligned}(a+b,12a,36a)&\sim(a+b,-3\cdot 2(a+b)+12a,9(a+b)-3\cdot 12a+36a)\\ &\sim(a+b,6(a-b),9(a+b)).\end{aligned}$$

By the above and (1.3) we obtain the result.

LEMMA 4.1 ([SW1, Lemma 4.1]). *Let $d$ be a discriminant and $n\in\mathbb{N}$. Then*

$$\sum_{m\mid n}\Big(\frac dm\Big)$$

$$=\begin{cases}\prod\limits_{(\frac dp)=1}(1+\text{ord}_p n) & \text{if } 2\mid \text{ord}_q n \text{ for every prime } q \text{ with } \big(\frac dq\big)=-1,\\ 0 & \text{otherwise,}\end{cases}$$

*where in the product $p$ runs over all distinct primes such that $p\mid n$ and $\big(\frac dp\big)=1$.*

THEOREM 4.2. *Let $n\in\mathbb{N}$. Then*

$$\Big|\Big\{\langle x,y\rangle\in\mathbb{Z}^2: n=\frac{3x^2-x}{2}+\frac{3y^2-y}{2}\Big\}\Big|=\sum_{k\mid 12n+1}(-1)^{(k-1)/2}$$

$$= \begin{cases} \prod_{p\equiv 1\ (\mathrm{mod}\ 4)} (1+\mathrm{ord}_p(12n+1)) \\ \qquad \text{if } 2 \mid \mathrm{ord}_q(12n+1) \text{ for every prime } q\equiv 3 \pmod 4, \\ 0 \quad \text{otherwise,} \end{cases}$$

*where* $p$ *runs over all primes satisfying* $p\equiv 1 \pmod 4$ *and* $p\mid(12n+1)$.

*Proof.* By Corollary 4.1 we have $R_{1,1}(1,3;n) = \frac{1}{2}R(2,0,18;24n+2) = \frac{1}{2}R(1,0,9;12n+1)$. Since $H(-36)=\{[1,0,9],[2,2,5]\}$ and $f(-36)=3$, by [SW1, Theorem 9.3] and Lemma 4.1, we have

$$\begin{aligned} &R(1,0,9;12n+1) \\ &\quad= \begin{cases} \left(1+\left(\frac{12n+1}{3}\right)\right) \prod_{(\frac{-4}{p})=1} (1+\mathrm{ord}_p(12n+1)) \\ \qquad \text{if } 2 \mid \mathrm{ord}_q(12n+1) \text{ for every prime } q \text{ with } \left(\frac{-4}{q}\right)=-1, \\ 0 \quad \text{otherwise} \end{cases} \\ &\quad= 2\sum_{k\mid(12n+1)} \left(\frac{-4}{k}\right), \end{aligned}$$

where $p$ runs over all primes satisfying $p\equiv 1 \pmod 4$ and $p\mid(12n+1)$. Thus the result follows.

THEOREM 4.3. *Let* $n\in\mathbb{N}$ *and* $8n+1=3^{\alpha}n_0$ $(3\nmid n_0)$. *Then*

$$\begin{aligned} \left|\left\{\langle x,y\rangle\in\mathbb{Z}^2 : n=\frac{3x^2-x}{2}+3y^2-y\right\}\right| &= \sum_{k\mid n_0}\left(\frac{-2}{k}\right) \\ &= \begin{cases} \prod_{p\equiv 1,3\ (\mathrm{mod}\ 8)} (1+\mathrm{ord}_p\, n_0) \\ \qquad \text{if } 2\mid \mathrm{ord}_q\, n_0 \text{ for every prime } q\equiv 5,7 \pmod 8, \\ 0 \quad \text{otherwise,} \end{cases} \end{aligned}$$

*where* $p$ *runs over all primes satisfying* $p\equiv 1,3 \pmod 8$ *and* $p\mid n_0$.

*Proof.* As $(1,4,12)\sim(1,0,8)$ and $(3,4,4)\sim(3,-2,3)$, by Theorem 4.1 we have

$$\begin{aligned} &2R_{1,2}(1,3;n) \\ &\quad= R(3,12,36;24n+3)-R(27,36,36;24n+3) \\ &\quad= R(1,4,12;8n+1)-R(9,12,12;8n+1) \\ &\quad= \begin{cases} R(1,4,12;8n+1)=R(1,0,8;8n+1) & \text{if } n\equiv 0,2 \pmod 3, \\ R(1,4,12;8n+1)-R(3,4,4;(8n+1)/3) & \\ \qquad = R(1,0,8;8n+1)-R(3,2,3;(8n+1)/3) & \text{if } n\equiv 1 \pmod 3. \end{cases} \end{aligned}$$

As $H(-32)=\{[1,0,8],[3,2,3]\}$ and $f(-32)=2$, by [SW1, Theorem 9.3] and Lemma 4.1, we see that for any odd positive integer $m$,

$$\begin{aligned} R(1,0,8;m) &= \begin{cases} \big(1+\big(\frac{-1}{m}\big)\big)\prod\limits_{(\frac{-8}{p})=1}(1+\mathrm{ord}_p\, m) \\ \qquad \text{if } 2\mid \mathrm{ord}_q\, m \text{ for every prime } q \text{ with } \big(\frac{-8}{q}\big)=-1, \\ 0 \quad \text{otherwise} \end{cases} \\ &= (1+(-1)^{(m-1)/2})\sum_{k\mid m}\Big(\frac{-8}{k}\Big) \end{aligned}$$

and

$$\begin{aligned} R(3,2,3;m) &= \begin{cases} \big(1-\big(\frac{-1}{m}\big)\big)\prod\limits_{(\frac{-8}{p})=1}(1+\mathrm{ord}_p\, m) \\ \qquad \text{if } 2\mid \mathrm{ord}_q\, m \text{ for every prime } q \text{ with } \big(\frac{-8}{q}\big)=-1, \\ 0 \quad \text{otherwise} \end{cases} \\ &= (1-(-1)^{(m-1)/2})\sum_{k\mid m}\Big(\frac{-8}{k}\Big), \end{aligned}$$

where $p$ runs over all primes satisfying $\big(\frac{-8}{p}\big)=1$ (i.e., $p\equiv 1,3 \pmod 8$) and $p\mid m$.

If $n\equiv 0,2 \pmod 3$, then $3\nmid (8n+1)$ and $n_0=8n+1$. By the above,

$$2R_{1,2}(1,3;n)=R(1,0,8;8n+1)=2\sum_{k\mid 8n+1}\Big(\frac{-8}{k}\Big)=2\sum_{k\mid 8n+1}\Big(\frac{-2}{k}\Big).$$

So the result is true. Now assume $n\equiv 1 \pmod 3$. From the above,

$$\begin{aligned} R_{1,2}(1,3;n) &= (R(1,0,8;8n+1)-R(3,2,3;(8n+1)/3))/2 \\ &= \sum_{k\mid 8n+1}\Big(\frac{-8}{k}\Big)-\sum_{k\mid \frac{8n+1}{3}}\Big(\frac{-8}{k}\Big)=\sum_{k\mid 8n+1,\, k\nmid \frac{8n+1}{3}}\Big(\frac{-8}{k}\Big) \\ &= \sum_{k\mid n_0}\Big(\frac{-8}{3^\alpha k}\Big)=\sum_{k\mid n_0}\Big(\frac{-8}{k}\Big). \end{aligned}$$

In view of Lemma 4.1 the theorem is proved.

Theorem 4.4. *Let $n\in\mathbb{N}$ and $4n+1=5^\alpha n_0=3^\beta n_1$ with $5\nmid n_0$ and $3\nmid n_1$. Then*

$$\Big|\Big\{\langle x,y\rangle\in\mathbb{Z}^2: n=\frac{3x^2-x}{2}+5\,\frac{3y^2-y}{2}\Big\}\Big|=\frac{1+\big(\frac{n_0}{5}\big)}{2}\sum_{k\mid n_1}\Big(\frac{-5}{k}\Big)$$

$$=\begin{cases} \prod_{p\equiv 1,3,7,9\ (\mathrm{mod}\ 20)}(1+\mathrm{ord}_p n_1) \\ \qquad \text{if } n_0\equiv \pm 1 \pmod 5 \text{ and } 2\mid \mathrm{ord}_q n_1 \text{ for} \\ \qquad \text{every prime } q\equiv 11,13,17,19 \pmod{20}, \\ 0 \quad \text{otherwise,} \end{cases}$$

*where* $p$ *runs over all primes satisfying* $p\equiv 1,3,7,9 \pmod{20}$ *and* $p\mid n_1$.

*Proof.* As $(1,2,6)\sim(1,0,5)$ and $(3,2,2)\sim(2,-2,3)$, by Theorem 4.1 we have

$$\begin{aligned} &2R_{1,5}(1,3;n) \\ &\quad = R(6,12,36;24n+6)-R(54,36,36;24n+6) \\ &\quad = R(1,2,6;4n+1)-R(9,6,6;4n+1) \\ &\quad = \begin{cases} R(1,2,6;4n+1)=R(1,0,5;4n+1) & \text{if } n\equiv 0,1 \pmod 3, \\ R(1,2,6;4n+1)-R(3,2,2;(4n+1)/3) & \\ \quad = R(1,0,5;4n+1)-R(2,2,3;(4n+1)/3) & \text{if } n\equiv 2 \pmod 3. \end{cases} \end{aligned}$$

As $H(-20)=\{[1,0,5],[2,2,3]\}$ and $f(-20)=1$, by [SW1, Theorem 9.3] and Lemma 4.1, we see that

$$\begin{aligned} &R(1,0,5;4n+1) \\ &\quad = \begin{cases} \left(1+\left(\frac{n_0}{5}\right)\right)\prod_{(\frac{-20}{p})=1}(1+\mathrm{ord}_p(4n+1)) \\ \qquad \text{if } 2\mid \mathrm{ord}_q(4n+1) \text{ for every prime } q \text{ with } \left(\frac{-20}{q}\right)=-1, \\ 0 \quad \text{otherwise} \end{cases} \\ &\quad = \left(1+\left(\frac{n_0}{5}\right)\right)\sum_{k\mid 4n+1}\left(\frac{-20}{k}\right), \end{aligned}$$

where $p$ runs over all primes satisfying $\left(\frac{-20}{p}\right)=1$ (that is, $p\equiv 1,3,7,9 \pmod{20}$) and $p\mid(4n+1)$.

If $n\equiv 0,1 \pmod 3$, then $3\nmid(4n+1)$ and $n_1=4n+1$. By the above we have

$$2R_{1,5}(1,3;n)=R(1,0,5;4n+1)=\left(1+\left(\frac{n_0}{5}\right)\right)\sum_{k\mid 4n+1}\left(\frac{-5}{k}\right).$$

So the result is true. Now assume $n\equiv 2 \pmod 3$. From [SW1, Theorem 9.3] and Lemma 4.1 we see that

$$R(2,2,3;(4n+1)/3) = \begin{cases} \big(1-\big(\frac{n_0/3}{5}\big)\big) \prod_{(\frac{-20}{p})=1} \big(1+\mathrm{ord}_p \frac{4n+1}{3}\big) \\ \quad \text{if } 2 \mid \mathrm{ord}_q \frac{4n+1}{3} \text{ for every prime } q \text{ with } \big(\frac{-20}{q}\big)=-1, \\ 0 \quad \text{otherwise} \end{cases} = \Big(1+\Big(\frac{n_0}{5}\Big)\Big) \sum_{k \mid \frac{4n+1}{3}} \Big(\frac{-20}{k}\Big),$$

where $p$ runs over all primes satisfying $\big(\frac{-20}{p}\big)=1$ and $p \mid \frac{4n+1}{3}$. Thus

$$\begin{aligned} R_{1,5}(1,3;n) &= (R(1,0,5;4n+1)-R(2,2,3;(4n+1)/3))/2 \\ &= \frac{1+\big(\frac{n_0}{5}\big)}{2}\Big(\sum_{k\mid 4n+1}\Big(\frac{-20}{k}\Big)-\sum_{k\mid\frac{4n+1}{3}}\Big(\frac{-20}{k}\Big)\Big) \\ &= \frac{1+\big(\frac{n_0}{5}\big)}{2}\sum_{k\mid n_1}\Big(\frac{-20}{3^\beta k}\Big) = \frac{1+\big(\frac{n_0}{5}\big)}{2}\sum_{k\mid n_1}\Big(\frac{-20}{k}\Big). \end{aligned}$$

This together with Lemma 4.1 completes the proof.

## 5. Formulas for $f_{a,b}\big(1,5;n+\frac{a+b}{5}\big)+f_{a,b}(3,5;n)$ when $5 \mid (a+b)$

LEMMA 5.1. *Let $n \in \mathbb{N}$ and $a,b,c \in \mathbb{Z}$ with $a,c>0$ and $b^2-4ac<0$. Then*

$$2\sum_{\substack{x,y\in\mathbb{Z},\, x\equiv 1,3 \,(\mathrm{mod}\, 10)\\ ax^2+bxy+cy^2=n}} 1 = R(a,b,c;n)-R(4a,2b,c;n)-R(25a,5b,c;n)+R(100a,10b,c;n).$$

*Proof.* Clearly

$$\begin{aligned} 2\sum_{\substack{x,y\in\mathbb{Z},\, x\equiv 1,3 \,(\mathrm{mod}\, 10)\\ ax^2+bxy+cy^2=n}} 1 &= \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv \pm1,\pm3 \,(\mathrm{mod}\, 10)\\ ax^2+bxy+cy^2=n}} 1 = \sum_{\substack{x,y\in\mathbb{Z},\, 2\nmid x\\ ax^2+bxy+cy^2=n}} 1 - \sum_{\substack{x,y\in\mathbb{Z},\, 2\nmid x,\, 5\mid x\\ ax^2+bxy+cy^2=n}} 1 \\ &= \sum_{\substack{x,y\in\mathbb{Z}\\ ax^2+bxy+cy^2=n}} 1 - \sum_{\substack{x,y\in\mathbb{Z},\, 2\mid x\\ ax^2+bxy+cy^2=n}} 1 - \sum_{\substack{x,y\in\mathbb{Z},\, 2\nmid x\\ 25ax^2+5bxy+cy^2=n}} 1 \\ &= R(a,b,c;n)-R(4a,2b,c;n)-(R(25a,5b,c;n)-R(100a,10b,c;n)). \end{aligned}$$

This proves the lemma.

THEOREM 5.1. *Let* $a,b,n\in\mathbb{N}$ *with* $5\mid(a+b)$. *Then*

$$\begin{aligned}f_{a,b}(1,5;n+(a+b)/5)+f_{a,b}(3,5;n)\\ &=\frac{1}{2}\big(R((a+b)/5,8a,80a;8n+9(a+b)/5)\\ &\quad-R(16a+(a+b)/5,72a,80a;8n+9(a+b)/5)\big).\end{aligned}$$

*Proof.* Set $n'=40n+9(a+b)$. From Theorem 2.1 and Lemma 5.1 we see that

$$\begin{aligned}&f_{a,b}(1,5;n+(a+b)/5)+f_{a,b}(3,5;n)\\ &=\sum_{\substack{x,y\in\mathbb{Z},\,x\equiv1,3\ (\mathrm{mod}\ 10)\\ (a+b)x^2+40axy+400ay^2=n'}}1-\sum_{\substack{x,y\in\mathbb{Z},\,x\equiv1,3\ (\mathrm{mod}\ 10)\\ (81a+b)x^2-360axy+400ay^2=n'}}1\\ &=\frac{1}{2}\big(R(a+b,40a,400a;n')-R(4(a+b),80a,400a;n')\\ &\quad-R(25(a+b),200a,400a;n')+R(100(a+b),400a,400a;n')\big)\\ &\quad-\frac{1}{2}\big(R(81a+b,-360a,400a;n')-R(4(81a+b),-720a,400a;n')\\ &\quad-R(25(81a+b),-1800a,400a;n')+R(100(81a+b),-3600a,400a;n')\big).\end{aligned}$$

For $a',b',c',k\in\mathbb{Z}$ we have $(a',b',c')\sim(a'+kb'+k^2c',b'+2kc',c')$. Thus

$$\begin{aligned}(4(81a+b),-720a,400a)&\sim(4(a+b),80a,400a),\\ (25(81a+b),-1800a,400a)&\sim(25(a+b),-200a,400a),\\ (100(81a+b),-3600a,400a)&\sim(100(a+b),-400a,400a).\end{aligned}$$

Hence applying (1.3) we have

$$\begin{aligned}R(4(81a+b),-720a,400a;n')&=R(4(a+b),80a,400a;n'),\\ R(25(81a+b),-1800a,400a;n')&=R(25(a+b),200a,400a;n'),\\ R(100(81a+b),-3600a,400a;n')&=R(100(a+b),400a,400a;n').\end{aligned}$$

Combining the above, we obtain

$$\begin{aligned}&f_{a,b}(1,5;n+(a+b)/5)+f_{a,b}(3,5;n)\\ &\qquad=\frac{1}{2}\big(R(a+b,40a,400a;n')-R(81a+b,-360a,400a;n')\big).\end{aligned}$$

This yields the result.

THEOREM 5.2. *Let* $a,c,n\in\mathbb{N}$ *with* $a<5c$, $(a,5c)\nmid n$ *and* $(a,5c)\nmid(n+c)$. *Then*

$$R(c,8a,80a;8n+9c)=R(16a+c,72a,80a;8n+9c).$$

*Proof.* Set $b = 5c - a$. Then $b \in \mathbb{N}$ and $(a, b) = (a, 5c - a) = (a, 5c)$. Thus $(a, b) \nmid n$ and $(a, b) \nmid (n + c)$. Hence $f_{a,b}(1, 5; n + c) = f_{a,b}(3, 5; n) = 0$ by (2.1) and Definition 2.2. Now Theorem 5.1 yields the result.

Corollary 5.1. *Let $k, m, n \in \mathbb{N}$ with $m < 20k$ and $2 \nmid n$. Then*

$$R(k, 2m, 20m; n + 9k) = R(k + 4m, 18m, 20m; n + 9k).$$

*Proof.* Putting $a = 2m$ and $c = 8k$ in Theorem 5.2, we see that $R(8k, 16m, 160m; 8n + 72k) = R(32m + 8k, 144m, 160m; 8n + 72k)$. This yields the result.

Remark 5.1. Let $m \in \{1, 2, \dots, 19\}$, $n \in \mathbb{N}$, $n \geq 10$ and $2 \mid n$. Putting $k = 1$ in Corollary 5.1 we have $R(1, 2m, 20m; n) = R(1 + 4m, 18m, 20m; n)$. As $(1, 2m, 20m) \sim (1, 0, m(20 - m))$ and $(1 + 4m, 18m, 20m) \sim (1 + 4m, 2m - 4, 4) \sim (4, 4 - 2m, 4m + 1)$ we see that

$$R(1, 0, m(20 - m); n) = R(4, 4 - 2m, 4m + 1; n). \tag{5.1}$$

For $m \leq 10$, (5.1) has been given in [SW2, Corollary 2.3].

Theorem 5.3. *Let $k, m, n \in \mathbb{N}$ with $k < 2m$. Then*

$$\begin{aligned} &f_{2k,4m-2k}(1, 4; n + m) + f_{2k,4m-2k}(3, 4; n) \\ &\quad = \frac{1}{2}\big(R(m, 16k, 128k; 8n + 9m) - R(m + 24k, 112k, 128k; 8n + 9m)\big). \end{aligned}$$

*Proof.* From Theorem 2.1 we see that

$$\begin{aligned} &f_{2k,4m-2k}(1, 4; n + m) + f_{2k,4m-2k}(3, 4; n) \\ &\quad = \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv 1,3 \pmod 8 \\ 4mx^2+64kxy+512ky^2=32n+36m}} 1 - \sum_{\substack{x,y\in\mathbb{Z},\, x\equiv 1,3 \pmod 8 \\ (96k+4m)x^2-448kxy+512ky^2=32n+36m}} 1 \\ &\quad = \frac{1}{2}\Big(\sum_{\substack{x,y\in\mathbb{Z},\, 2\nmid x \\ mx^2+16kxy+128ky^2=8n+9m}} 1 - \sum_{\substack{x,y\in\mathbb{Z},\, 2\nmid x \\ (24k+m)x^2-112kxy+128ky^2=8n+9m}} 1\Big) \\ &\quad = \frac{1}{2}\big(R(m, 16k, 128k; 8n + 9m) - R(4m, 32k, 128k; 8n + 9m) \\ &\qquad - R(24k + m, -112k, 128k; 8n + 9m) \\ &\qquad + R(96k + 4m, -224k, 128k; 8n + 9m)\big). \end{aligned}$$

As $(96k + 4m, -224k, 128k) \sim (128k, 224k, 96k + 4m) \sim (128k, -32k, 4m) \sim (4m, 32k, 128k)$, we have

$$R(96k + 4m, -224k, 128k; 8n + 9m) = R(4m, 32k, 128k; 8n + 9m).$$

Thus the result follows.

COROLLARY 5.2. *Let* $k, m, n \in \mathbb{N}$ *with* $k < 2m$, $(k, 2m) \nmid (n+m)$ *and* $(k, 2m) \nmid n$. *Then*

$$R(m, 16k, 128k; 8n+9m) = R(m+24k, 112k, 128k; 8n+9m).$$

*Proof.* As $(k, 2m-k) = (k, 2m)$, by (2.1) and Definition 2.2 we have $f_{2k,4m-2k}(1, 4; n+m) = f_{2k,4m-2k}(3, 4; n) = 0$. Now applying Theorem 5.3 we deduce the result.

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