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FIVE CONGRUENCES FOR PRIMES

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1.Introduction.

Let p be an odd prime. In 1988, using the formula for the sum $\sum_{k\equiv r \pmod 8}\binom{n}{k}$ the author proved that (cf.[7], Theorem 2.6)

$$\sum_{1\leqslant k<\frac{p}{2}}\frac{2^k}{k}\equiv 4(-1)^{\frac{p-1}{2}}\sum_{1\leqslant k\leqslant\frac{p+1}{4}}\frac{(-1)^{k-1}}{2k-1} \pmod p$$

and

$$\sum_{1\leqslant k<\frac{p}{2}}\frac{1}{k\cdot 2^k}\equiv -4\sum_{\frac{1+(-1)^{\frac{p-1}{2}}}{2}\leqslant k<\frac{p}{8}}\frac{1}{4k-(-1)^{\frac{p-1}{2}}} \pmod p.$$

In 1995, using a similar method Zhi-Wei Sun [9] proved the author's conjecture

$$\sum_{1\leqslant k<\frac{p}{2}}\frac{1}{k\cdot 2^k}\equiv\sum_{1\leqslant k<\frac{3p}{4}}\frac{(-1)^{k-1}}{k} \pmod p.$$

Later, Zun Shan and Edward T.H. Wang [5] gave a simple proof of the above congruence. In [9] and [10], Zhi-Wei Sun also pointed out another congruence

$$\sum_{1\leqslant k<\frac{p}{2}}\frac{3^k}{k}\equiv\sum_{1\leqslant k<\frac{p}{6}}\frac{(-1)^k}{k} \pmod p.$$

In this paper, by using the formulas for Fibonacci quotient and Pell quotient we obtain

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX the following five congruences:

$$\sum_{1\leqslant k<\frac{p}{2}} \frac{2^k}{k} \equiv 2 \sum_{\frac{p}{4}<k<\frac{p}{2}} \frac{(-1)^{k-1}}{k} \pmod p, \tag{1.1}$$

$$\sum_{1\leqslant k<\frac{p}{2}} \frac{5^k}{k} \equiv 2 \sum_{\frac{p}{5}<k<\frac{p}{2}} \frac{(-1)^{k-1}}{k} \pmod p, \tag{1.2}$$

$$\sum_{1\leqslant k<\frac{p}{2}} \frac{2^k}{k} \equiv - \sum_{\frac{p}{8}<k<\frac{3p}{8}} \frac{1}{k} \pmod p, \tag{1.3}$$

$$\sum_{1\leqslant k<\frac{p}{2}} \frac{1}{k\cdot 2^k} \equiv - \sum_{\frac{p}{4}<k<\frac{3p}{8}} \frac{1}{k} \pmod p, \tag{1.4}$$

$$\sum_{1\leqslant k<\frac{p}{2}} \frac{3^k}{k} \equiv - \sum_{\frac{p}{12}<k<\frac{p}{6}} \frac{1}{k} \pmod p, \tag{1.5}$$

where $p > 5$ is a prime.

2. Basic Lemmas.

The Lucas sequences $\{u_n(a,b)\}$ and $\{v_n(a,b)\}$ are defined as follows:

$$u_0(a,b) = 0,\ u_1(a,b) = 1,\ u_{n+1}(a,b) = bu_n(a,b) - au_{n-1}(a,b) \quad (n \geq 1),$$
$$v_0(a,b) = 2,\ v_1(a,b) = b,\ v_{n+1}(a,b) = bv_n(a,b) - av_{n-1}(a,b) \quad (n \geq 1).$$

It is well known that

$$u_n(a,b) = \frac{1}{\sqrt{b^2-4a}}\Big\{\Big(\frac{b+\sqrt{b^2-4a}}{2}\Big)^n - \Big(\frac{b-\sqrt{b^2-4a}}{2}\Big)^n\Big\} \quad (b^2-4a \neq 0)$$

and

$$v_n(a,b) = \Big(\frac{b+\sqrt{b^2-4a}}{2}\Big)^n + \Big(\frac{b-\sqrt{b^2-4a}}{2}\Big)^n.$$

Let p be an odd prime, and let m be an integer with $m \not\equiv 0 \pmod p$. It is evident that

$$2\sum_{\substack{k=1\\ 2\nmid k}}^{p-1} \binom{p}{k}(\sqrt{m})^k = (1+\sqrt{m})^p - (1-\sqrt{m})^p - 2(\sqrt{m})^p$$

and

$$2\sum_{\substack{k=1\\ 2\mid k}}^{p-1} \binom{p}{k}(\sqrt{m})^k = (1+\sqrt{m})^p + (1-\sqrt{m})^p - 2.$$

Since

$$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1} \equiv \frac{(-1)^{k-1}}{k}p \pmod{p^2},$$

by the above one can easily prove

Lemma 1([7], Lemma 2.4). *Suppose that p is an odd prime and that m is an integer such that $p\nmid m$. Then*

$$\text{(i)}\quad \sum_{k=1}^{(p-1)/2}\frac{1}{k\cdot m^k}\equiv\frac{m^{p-1}-1}{p}-2\cdot\frac{\left(\frac{m}{p}\right)u_p(1-m,2)-1}{p}\pmod p,$$

$$\text{(ii)}\quad \sum_{k=1}^{(p-1)/2}\frac{m^k}{k}\equiv\frac{2-v_p(1-m,2)}{p}\pmod p,$$

where $\left(\frac{m}{p}\right)$ is the Legendre symbol.

For any odd prime p and integer m set $q_p(m)=\frac{m^{p-1}-1}{p}$. Using Lemma 1 we can prove

Proposition 1. *Let m be an integer, and p be an odd prime such that $p\nmid m(m-1)$. Then*

$$\begin{aligned}\frac{u_{p-(\frac{m}{p})}(1-m,2)}{p}&\equiv\frac{(m-2)\left(\frac{m}{p}\right)-m}{4m}\Big(\sum_{k=1}^{(p-1)/2}\frac{m^k}{k}+q_p(m-1)\Big)\\&\equiv\frac{(m-2)\left(\frac{m}{p}\right)-m}{4}\Big(\sum_{k=1}^{(p-1)/2}\frac{1}{k\cdot m^k}+q_p(m-1)-q_p(m)\Big)\pmod p.\end{aligned}$$

Proof. Set $u_n=u_n(1-m,2)$ and $v_n=v_n(1-m,2)$. From [1],[4] and [6, Lemma 1.7] we know that

$$v_n^2-4mu_n^2=4(1-m)^n,\ v_n=2u_{n+1}-2u_n,\ u_n=\frac{1}{2m}(v_{n+1}-v_n)$$

and

$$u_{p-(\frac{m}{p})}\equiv u_p-\Big(\frac{m}{p}\Big)\equiv 0\pmod p.$$

Thus,

$$v^2_{p-(\frac{m}{p})}\equiv 4(1-m)^{p-\left(\frac{m}{p}\right)}\pmod{p^2}$$

and hence

$$v_{p-(\frac{m}{p})}\equiv\pm 2\Big(\frac{1-m}{p}\Big)(1-m)^{(p-(\frac{m}{p}))/2}\pmod{p^2}.$$

If $\left(\frac{m}{p}\right)=1$ then $v_{p-1}=2u_p-2u_{p-1}\equiv 2\pmod p$. Hence, by the above we get

$$v_{p-1}\equiv 2(1-m)^{(p-1)/2}\Big(\frac{1-m}{p}\Big)\equiv 2+q_p(m-1)p\pmod{p^2}. \tag{2.1}$$

Now applying Lemma 1 we find

$$\begin{aligned}\frac{u_{p-1}}{p}&=\frac{1}{2m}\cdot\frac{v_p-v_{p-1}}{p}=\frac{1}{2m}\Big(\frac{v_p-2}{p}-\frac{v_{p-1}-2}{p}\Big)\\&\equiv\frac{1}{2m}\Big(-\sum_{k=1}^{(p-1)/2}\frac{m^k}{k}-q_p(m-1)\Big)\pmod p\end{aligned}$$

and

$$\frac{u_{p-1}}{p} = \frac{2u_p - v_{p-1}}{2p} = \frac{u_p - 1}{p} + \frac{1}{2} \cdot \frac{2 - v_{p-1}}{p}$$
$$\equiv \frac{1}{2}\Big(q_p(m) - \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot m^k} - q_p(m-1)\Big) \pmod p.$$

This proves the result in the case $\left(\frac{m}{p}\right) = 1$.

If $\left(\frac{m}{p}\right) = -1$ then

$$v_{p+1} = 2u_{p+1} - 2(1-m)u_p \equiv 2(1-m) \pmod p.$$

So

$$v_{p+1} \equiv 2(1-m)\Big(\frac{1-m}{p}\Big)(1-m)^{(p-1)/2} \equiv (1-m)(2 + q_p(m-1)p) \pmod{p^2}. \qquad (2.2)$$

Note that

$$u_{p+1} = \frac{1}{2m}(v_{p+1} + (m-1)v_p) = \frac{1}{2}v_{p+1} + (1-m)u_p.$$

Applying (2.2) and Lemma 1, one can easily deduce the desired result. Hence the proof is complete.

Corollary 1. *Let p be an odd prime, and $\{P_n\}$ denote the Pell sequence given by $P_0 = 0,\ P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$ $(n \geq 1)$. Then*

(i) $$\sum_{k=1}^{(p-1)/2} \frac{2^k}{k} \equiv -4\frac{P_{p-\left(\frac{2}{p}\right)}}{p} \pmod p.$$

(ii) $$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv -2\frac{P_{p-\left(\frac{2}{p}\right)}}{p} + q_p(2) \pmod p.$$

Proof. Taking $m = 2$ in Proposition 1 gives the result.

Corollary 2. *Let $p > 3$ be a prime, $S_0 = 0,\ S_1 = 1$ and $S_{n+1} = 4S_n - S_{n-1}$ $(n \geq 1)$. Then*

(i) $$\sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv -3\Big(\frac{3}{p}\Big)\frac{S_{p-\left(\frac{3}{p}\right)}}{p} - q_p(2) \pmod p.$$

(ii) $$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 3^k} \equiv -\Big(\frac{3}{p}\Big)\frac{S_{p-\left(\frac{3}{p}\right)}}{p} - q_p(2) + q_p(3) \pmod p.$$

Proof. Suppose a and b are integers. From [4] we know that

$$u_{2n}(a,b) = u_n(a,b)v_n(a,b)$$

and

$$u_{p-(\frac{b^2-4a}{p})}(a,b) \equiv u_p(a,b) - (\frac{b^2-4a}{p}) \equiv 0 \pmod p.$$

Thus,

$$v_{p-(\frac{3}{p})}(-2,2) = \begin{cases} 2u_p(-2,2) - 2u_{p-1}(-2,2) \equiv 2 \pmod p & \text{if } \left(\frac{3}{p}\right) = 1, \\ 2u_{p+1}(-2,2) + 4u_p(-2,2) \equiv -4 \pmod p & \text{if } \left(\frac{3}{p}\right) = -1 \end{cases}$$
$$\equiv 3(\frac{3}{p}) - 1 \pmod p.$$

Observing that $S_n = u_n(1,4) = 2^{-n}u_{2n}(-2,2)$ we get

$$\begin{aligned} S_{p-(\frac{3}{p})}/p &= 2^{\left(\frac{3}{p}\right)-p} v_{p-(\frac{3}{p})}(-2,2) u_{p-(\frac{3}{p})}(-2,2)/p \\ &\equiv 2^{\left(\frac{3}{p}\right)-1}\left(3(\frac{3}{p}) - 1\right) u_{p-(\frac{3}{p})}(-2,2)/p \\ &= \frac{1}{2}\left(1 + 3(\frac{3}{p})\right) u_{p-(\frac{3}{p})}(-2,2)/p \pmod p. \end{aligned}$$

This together with the case $m = 3$ of Proposition 1 gives the result.

Remark 1. The sequence $\{S_n\}$ was first introduced by my brother Zhi-Wei Sun, who gave the formula for the sum $\sum_{k \equiv r \pmod{12}} \binom{n}{k}$ in terms of $\{S_n\}$ (cf.[10]).

Corollary 3. *Let $p > 5$ be a prime, and $\{F_n\}$ denote the Fibonacci sequence. Then*

(i) $$\sum_{k=1}^{(p-1)/2} \frac{5^k}{k} \equiv -5 \frac{F_{p-\left(\frac{5}{p}\right)}}{p} - 2q_p(2) \pmod p.$$

(ii) $$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 5^k} \equiv -\frac{F_{p-\left(\frac{5}{p}\right)}}{p} + q_p(5) - 2q_p(2) \pmod p.$$

Proof. It is easily seen that $u_n(-4,2) = 2^{n-1}F_n$. So we have

$$\frac{F_{p-\left(\frac{5}{p}\right)}}{p} = 2^{1-p+\left(\frac{5}{p}\right)} \frac{u_{p-\left(\frac{5}{p}\right)}(-4,2)}{p} \equiv 2^{\left(\frac{5}{p}\right)} \frac{u_{p-\left(\frac{5}{p}\right)}(-4,2)}{p} \pmod p.$$

Combining this with the case $m = 5$ of Proposition 1 yields the result.

Let $\{B_n\}$ and $\{B_n(x)\}$ be the Bernoulli numbers and Bernoulli polynomials given by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n \geq 2)$$

and

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

It is well known that (cf.[3])

$$\sum_{x=0}^{n-1} x^m = \frac{1}{m+1}\left(B_{m+1}(n) - B_{m+1}\right).$$

Lemma 2. *Let p be an odd prime, and let m be a positive integer such that $p \nmid m$. If $s \in \{1, 2, \dots, m-1\}$ then*

$$\sum_{1 \leqslant k \leqslant [\frac{sp}{m}]} \frac{1}{k} \equiv -\Big(B_{p-1}\Big(\{\frac{sp}{m}\}\Big) - B_{p-1}\Big) \pmod p,$$

where $[x]$ is the greatest integer not exceeding x and $\{x\} = x - [x]$.

Proof. Clearly,

$$\sum_{1 \leqslant k \leqslant [\frac{sp}{m}]} \frac{1}{k} \equiv \sum_{1 \leqslant k \leqslant [\frac{sp}{m}]} k^{p-2} = \frac{1}{p-1}\Big(B_{p-1}\Big([\frac{sp}{m}]+1\Big) - B_{p-1}\Big)$$
$$= \frac{1}{p-1}\Big(B_{p-1}\Big(\frac{sp}{m}+1-\{\frac{sp}{m}\}\Big) - B_{p-1}\Big) \pmod p.$$

For any rational p-integers x and y it is evident that (cf.[3])

$$pB_k(x) = \sum_{r=0}^{k} \binom{k}{r} pB_r x^{k-r} \equiv 0 \pmod p \quad \text{for} \quad k = 0, 1, \dots, p-2$$

and so

$$B_{p-1}(x+py) - B_{p-1}(x) = \sum_{k=0}^{p-2} \binom{p-1}{k} B_k(x)(py)^{p-1-k} \equiv 0 \pmod p.$$

Hence, by the above and the relation $B_n(1-x) = (-1)^n B_n(x)$ (cf.[3]) we get

$$\sum_{1 \leqslant k \leqslant [\frac{sp}{m}]} \frac{1}{k} \equiv \frac{1}{p-1}\Big(B_{p-1}(1-\{\frac{sp}{m}\}) - B_{p-1}\Big) \equiv -\Big(B_{p-1}(\{\frac{sp}{m}\}) - B_{p-1}\Big) \pmod p.$$

This proves the lemma.

3. Proof of (1.1)–(1.5).

In [8], using the formula for the sum $\sum_{k \equiv r \pmod 8} \binom{n}{k}$ the author proved that

$$\frac{P_{p-(\frac{2}{p})}}{p} \equiv \frac{1}{2} \sum_{\frac{p}{4}<k<\frac{p}{2}} \frac{(-1)^k}{k} \pmod p \tag{3.1}$$

and

$$\frac{P_{p-(\frac{2}{p})}}{p} \equiv \frac{1}{4} \sum_{\frac{p}{8}<k<\frac{3p}{8}} \frac{1}{k} \pmod p. \tag{3.2}$$

Here, (3.1) was found by Zhi-Wei Sun[10], and (3.2) was also given by H.C.Williams [12].

Now, putting (3.1) and (3.2) together with Corollary 1(i) proves (1.1) and (1.3).

To prove (1.2), we note that H.C.Williams (cf.[11]) has shown that

$$\frac{F_{p-(\frac{5}{p})}}{p} \equiv -\frac{2}{5}\sum_{k=1}^{p-1-[p/5]}\frac{(-1)^{k-1}}{k} \pmod p.$$

Since Eisenstein it is well known that (cf.[6])

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^{k-1}}{k} \equiv \frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k} \equiv q_p(2) \pmod p.$$

Thus, by William's result,

$$\begin{aligned}\frac{F_{p-(\frac{5}{p})}}{p} &\equiv -\frac{2}{5}\Big(2q_p(2) - \sum_{k=1}^{[p/5]}\frac{(-1)^{k-1}}{k}\Big)\\ &\equiv -\frac{2}{5}\Big(q_p(2) + \sum_{\frac{p}{5}<k<\frac{p}{2}}\frac{(-1)^{k-1}}{k}\Big) \pmod p.\end{aligned}$$

Hence, by Corollary 3(i) we have

$$\sum_{1\leqslant k<\frac{p}{2}}\frac{5^k}{k} \equiv -5\frac{F_{p-(\frac{5}{p})}}{p} - 2q_p(2) \equiv 2\sum_{\frac{p}{5}<k<\frac{p}{2}}\frac{(-1)^{k-1}}{k} \pmod p.$$

This proves (1.2).

Now consider (1.4). From [2] we know that

$$B_{p-1}(\{\frac{p}{4}\}) - B_{p-1} \equiv 3q_p(2) \qquad \pmod p$$

and

$$B_{p-1}(\{\frac{3p}{8}\}) - B_{p-1} \equiv -2\frac{P_{p-(\frac{2}{p})}}{p} + 4q_p(2) \pmod p.$$

Thus, by using Lemma 2 we obtain

$$\begin{aligned}-\sum_{\frac{p}{4}<k<\frac{3p}{8}}\frac{1}{k} &= \sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k} - \sum_{1\leqslant k<\frac{3p}{8}}\frac{1}{k}\\ &\equiv -(B_{p-1}(\{\frac{p}{4}\}) - B_{p-1}) + B_{p-1}(\{\frac{3p}{8}\}) - B_{p-1}\\ &\equiv -3q_p(2) + 4q_p(2) - 2\frac{P_{p-(\frac{2}{p})}}{p} \pmod p.\end{aligned}$$

This together with Corollary 1(ii) proves (1.4).

Finally we consider (1.5). By [2],

$$B_{p-1}(\{\frac{p}{6}\}) - B_{p-1} \equiv 2q_p(2) + \frac{3}{2}q_p(3) \pmod p$$

and

$$B_{p-1}(\{\frac{p}{12}\})-B_{p-1}\equiv 3\big(\frac{3}{p}\big)\frac{S_{p-\left(\frac{3}{p}\right)}}{p}+3q_p(2)+\frac{3}{2}q_p(3) \pmod p.$$

Thus, by Lemma 2 and Corollary 2(i),

$$\begin{aligned}-\sum_{\frac{p}{12}<k<\frac{p}{6}}\frac{1}{k}&\equiv (B_{p-1}(\{\frac{p}{6}\})-B_{p-1})-(B_{p-1}(\{\frac{p}{12}\})-B_{p-1})\\&\equiv 2q_p(2)+\frac{3}{2}q_p(3)-3q_p(2)-\frac{3}{2}q_p(3)-3\big(\frac{3}{p}\big)\frac{S_{p-\left(\frac{3}{p}\right)}}{p}\\&\equiv \sum_{1\leqslant k<\frac{p}{2}}\frac{3^k}{k} \pmod p.\end{aligned}$$

This proves (1.5) and the proof is complete.

Remark 2. The congruences (1.1)–(1.3) can also be proved by using the method in the proof of (1.4) or (1.5).

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